

Lecture 2

Existence Proofs: Nash, Brouwer, and Sperner

Instructor: Prof. Constantinos Daskalakis

In this lecture, we will do a deep dive into the proof of Brouwer's fixed point theorem, the main theorem that we invoked in the previous lecture to show the existence of Nash equilibrium. We will provide an elementary proof of Brouwer's theorem, one of several in the literature, with the goal of distilling Brouwer's existence-of-fixed-points result into a pure, combinatorial form. In particular, we seek to provide an answer to the following question:

What is the combinatorial essence of Brouwer's fixed point existence theorem?

Towards an answer, we will provide a proof of Brouwer's theorem via another existence theorem, known as Sperner's lemma. This pertains to a combinatorial structure, namely colorings of a triangulated grid, and it certifies the guaranteed existence of certain colorful triangles in this grid, if the coloring of the grid's boundary meets certain conditions. Given that Sperner's lemma pertains to combinatorial structures, by showing Brouwer's theorem through Sperner's lemma we come closer to a satisfying answer to the above question. Distilling even further, we will provide a proof of Sperner's lemma as a corollary of an elementary existence result, namely a parity argument in directed graphs. All in all, at the end of this lecture, we will have a surprisingly crisp answer to our question:

Brouwer's fixed point theorem is a corollary of the fact that any directed graph has an even number of odd-degree vertices.

And why are we interested in this pursuit? One reason is that we want to de-mystify what makes Nash equilibria exist in every game, and what makes fixed points exist in every continuous function from a convex compact set to itself.

Another reason is that designing algorithms for computing Nash equilibria can benefit from understanding the nature of the combinatorial argument underlying the existence of Nash equilibria. Indeed, we will soon see algorithms for computing Nash equilibria using the findings from this lecture.

And, in the reverse direction, understanding whether there are complexity barriers in the computation of Nash equilibria might benefit from known barriers for computing Brouwer fixed points, and colorful triangles in colored grids. Indeed, we will develop these ideas to study the computational complexity of Nash equilibria.

Relating the last two points, in *this* lecture we will lay the foundations for proving that: computing Nash equilibria can be polynomial-time reduced to computing fixed points of Lipschitz continuous functions; that the latter can be polynomial-time reduced to finding colorful triangles, guaranteed to exist in some large, colored grid by Sperner's lemma; and that the latter can be polynomial-time reduced to finding odd degree vertices in some large, directed graph given another odd degree vertex in that graph. So, informally, Nash will reduce to Brouwer which will reduce to Sperner which will reduce to a computational problem capturing the parity argument in directed graphs. This direction of reductions will give us ideas for Nash equilibrium computation algorithms.

Surprisingly, in a *future* lecture we will also establish reductions in the *reverse* direction, namely that parity reduces to Sperner which reduces to Brouwer which reduces to Nash! And we will use this direction of reductions to study the computational complexity of Nash equilibrium computation.

L2.1 Sperner's lemma

Key to distilling the combinatorial essence of why Nash equilibria exist in every game and why fixed points exist in every continuous function mapping a convex compact set to itself is a theorem that, at first glance, has little to do with fixed points nor equilibria: Sperner's lemma.

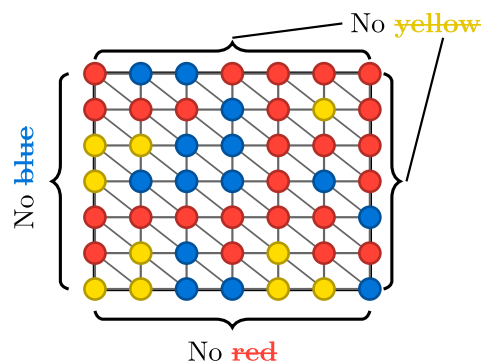


Figure 1: A Sperner coloring.

Sperner's lemma applies to a 3-colored $N \times N$ grids of points. There are high-dimensional analogues of the lemma, but we will stick to 2 dimensions in this lectures. For the lemma to apply, the rules are simple. Each point is colored with one of three colors: red, blue, or yellow. The coloring must, however, satisfy the following *boundary* conditions:

- (i) The left column cannot contain any blue;
- (ii) The bottom row cannot contain any red;
- (iii) The right column and top row cannot contain any yellow.

Any coloring that satisfies these conditions is called a *Sperner coloring*. An example of a coloring satisfying these rules is shown in Figure 1.

Given any Sperner coloring, we are interested in finding a trichromatic triangle, that is, a triangular cell whose vertices are colored red, blue, and yellow. Sperner's lemma guarantees that such a cell is guaranteed to exist, no matter the Sperner coloring.

Theorem L2.1 (Sperner’s lemma). Consider a Sperner coloring of a triangulated grid of any size. There must exist at least *one* trichromatic triangle.

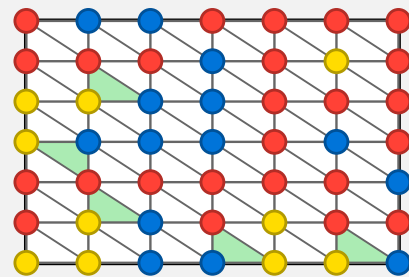
In fact, there must exist an *odd* number of trichromatic triangles.

We illustrate the previous theorem in the colored grid of Figure 1.

Example L2.2.

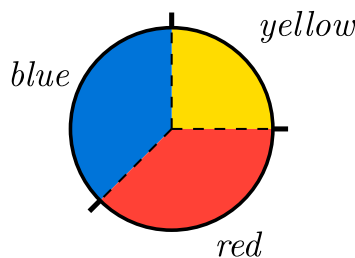
In the coloring of Figure 1, there are a total of *five* trichromatic triangles, as highlighted in green on the right.

Indeed, five is an odd number, validating the statement of Theorem L2.1.



L2.1.1 The connection between Brouwer and Sperner

What does Sperner’s lemma have to do with Brouwer’s fixed point theorem? The connection is not immediate, but upon second thought, several glimpses of connections emerge. For one, both results are existence results. Furthermore, both results are trivially false if the “boundary conditions” in their statements are violated. In the case of Brouwer’s fixed point theorem, the boundary conditions are that the continuous function must map the compact set to itself, i.e. the boundary points must be mapped to somewhere inside the set. In the case of Sperner’s lemma, the boundary conditions are the coloring rules on the boundary.

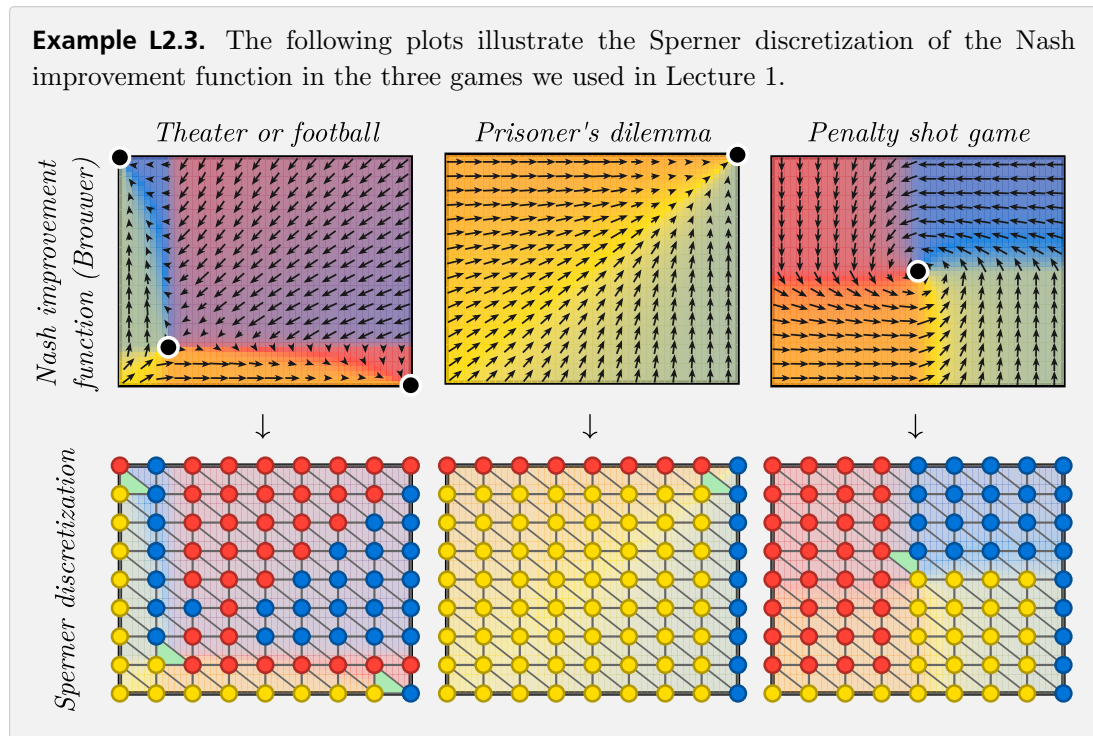


As it turns out, Sperner’s lemma can be viewed as a “discretized version” of Brouwer’s fixed point theorem. To see the connection, consider a continuous function mapping $[0, 1]^2$ to itself and, depending on the direction of $f(z) - z$, assign red, yellow, or blue to the vertices of a fine, triangulated grid, whose boundary matches that of $[0, 1]^2$, according to the rules shown on the left. (This coloring is a more boring version of the coloring we used in Lecture 1, but it will work for our purposes.)

If the direction of $f(z) - z$ lies in the yellow-blue, blue-red, or yellow-red boundary, we can assign any one of the two compatible colors, but we will make sure that, for grid points lying on the boundary of $[0, 1]^2$, we will break ties in favor of the color that does not violate the Sperner coloring conditions. Because f maps $[0, 1]^2$ to itself, there should always be at least one such option! Thus, the coloring we will obtain will be a valid Sperner coloring, and it will have at least one trichromatic triangle.

In turn, it should be intuitively clear why trichromatic triangles have value vis-à-vis the fixed point behavior of f : they are triangles where $f(z) - z$ changes direction within a small distance and, due to continuity, $f(z) - z$ can't be too large.

We formalize these ideas in the next sections, arriving at two results. First, we will show a complete proof of Brouwer's fixed point theorem, using Sperner's lemma and a compactness argument. Second, we will establish the following computational reduction. Suppose we are given access to an algorithm that takes as input a Sperner coloring of a triangulated grid and computes a trichromatic triangle guaranteed by Sperner's lemma. Then, we can use this algorithm to compute approximate Brouwer fixed points of a Lipschitz continuous function, f , from $[0, 1]^2$ to itself, by discretizing the domain into a grid whose cells have small enough diameter, as a function of the Lipschitz constant and the desired approximation, coloring the vertices of this grid according to the scheme presented above, and finding a trichromatic triangle. We illustrate how this reduction would work with an example.



It is worth noting that the trichromatic triangles obtained via the above reduction are not always in the proximity of exact fixed points of the function. Unless the discretization is fine enough and f has extra properties, we will only guarantee that the trichromatic triangles are in the proximity of approximate fixed points. While this is not the case in the examples above, it can be the case.

L2.1.2 Formalizing the connection

To make the argument formal, we need to establish a formal connection between a trichromatic triangle and an approximate Brouwer fixed point, and connect that to a choice of discretization parameter.

By the Heine-Cantor theorem, any continuous function f on a compact set is *uniformly* continuous, which implies that:

$$\forall \epsilon > 0, \exists \delta(\epsilon) > 0 : \quad \left\| z - w \right\|_{\infty} < \delta(\epsilon) \implies \left\| f(z) - f(w) \right\|_{\infty} < \epsilon. \quad (1)$$

Now, given f and ϵ , consider a triangulation of $[0, 1]^2$ in which the diameter of every triangle is δ in ℓ_{∞} . Assign colors to the vertices of the triangulation according to the direction of $f(z) - z$, using the coloring scheme discussed above, and breaking ties in an arbitrary way but respecting the Sperner coloring conditions. Let us call the resulting coloring a *Sperner discretization of f of diameter δ* . Then, the following approximation bound can be established.

Theorem L2.4. Suppose that z_Y is the yellow corner of a trichromatic triangle in a Sperner discretization of some continuous function $f : [0, 1]^2 \rightarrow [0, 1]^2$ of diameter $\delta \leq \delta(\epsilon)$, where $\delta(\epsilon)$ satisfies 1 for some ϵ . Then

$$\left\| f(z_Y) - z_Y \right\|_{\infty} < \epsilon + \delta.$$

Proof. Let z_R, z_B , and z_Y be the red, blue, and yellow vertices of the trichromatic triangle. The key observation is that, by the coloring rule:

- $(f(z_Y) - z_Y)_x$ and $(f(z_B) - z_B)_x$ have opposite signs if they are nonzero
- $(f(z_Y) - z_Y)_y$ and $(f(z_R) - z_R)_y$ have opposite signs if they are non-zero

Thus, we can write

$$\begin{aligned} |(f(z_Y) - z_Y)_x| &\leq |(f(z_Y) - z_Y)_x - (f(z_B) - z_B)_x| \\ &\leq |(f(z_Y) - f(z_B))_x - (z_Y - z_B)_x| \\ &\leq \left\| f(z_Y) - f(z_B) \right\|_{\infty} + \left\| z_Y - z_B \right\|_{\infty} < \epsilon + \delta. \end{aligned}$$

and similarly

$$\begin{aligned} |(f(z_Y) - z_Y)_y| &\leq |(f(z_Y) - z_Y)_y - (f(z_R) - z_R)_y| \\ &\leq |(f(z_Y) - f(z_R))_y - (z_Y - z_R)_y| \\ &\leq \left\| f(z_Y) - f(z_R) \right\|_{\infty} + \left\| z_Y - z_R \right\|_{\infty} < \epsilon + \delta. \end{aligned}$$

From here, we can just use the definition of infinity norm:

$$\left\| f(z_Y) - z_Y \right\|_{\infty} = \max \left\{ |(f(z_Y) - z_Y)_x|, |(f(z_Y) - z_Y)_y| \right\} < \epsilon + \delta.$$

□

Corollary L2.5. Consider the same setup as before, but now choose $\delta := \min\{\delta(\epsilon), \epsilon\}$ for a given $\epsilon > 0$. Then z_Y is a 2ϵ -approximate fixed point of f , i.e.

$$\|f(z_Y) - z_Y\|_{\infty} < 2\epsilon.$$

In turn, using a standard compactness argument, Corollary L2.5 implies Brouwer's fixed point theorem for continuous functions from $[0, 1]^2$ to itself.

Corollary L2.6 (Brouwer's fixed point theorem, unit square). Any continuous function from $[0, 1]^2$ to itself has a fixed point.

Proof. Consider the sequence of approximation parameters $\epsilon_i := 2^{-i}$ for $i \in \mathbb{N}_{\geq 1}$, and the corresponding discretization parameters $\delta_i := \min\{\delta(\epsilon_i), \epsilon_i\}$, as in Corollary L2.5. For each i , we color the vertices of the resulting triangulation as above, so that they satisfy the conditions of Sperner's lemma, and identify a trichromatic triangle which is then guaranteed to exist. Let us denote by $z_{Y,i}$ the yellow vertex of that triangle, which satisfies $\|f(z_{Y,i}) - z_{Y,i}\| < 2\epsilon_i$. Now consider the sequence of points $(z_{Y,i})_i$. Since $z_{Y,i} \in [0, 1]^2$ for all i , and $[0, 1]^2$ is a compact set, there exists a convergent subsequence $(z_{Y,n_j})_j$; let z_Y^* denote the limit of this subsequence. By the continuity of f , the function $d(z) := \|f(z) - z\|_{\infty}$ is also continuous. Hence,

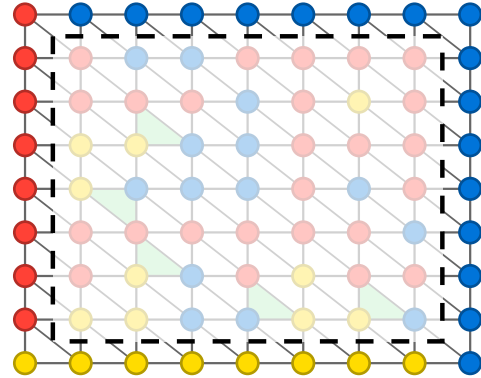
$$d(z_Y^*) = \lim_{j \rightarrow \infty} d(z_{Y,n_j}).$$

Since $d(z_{Y,n_j}) \in [0, 2 \cdot 2^{-j}]$, we conclude $d(z_Y^*) = 0$, which is equivalent to $f(z_Y^*) = z_Y^*$. This proves that a fixed point exists. $\square$

L2.2 Proof of Sperner's lemma

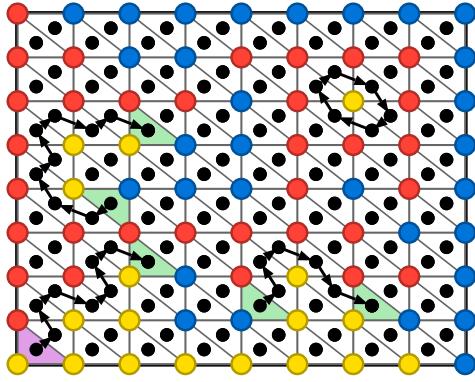
Now we turn to proving Sperner's lemma. As it turns out, the lemma can be obtained as a corollary of a very basic parity argument on directed graphs.

Before jumping into the proof, let us make our life simpler. Without loss of generality, we will assume that, at the boundary of the grid, the Sperner coloring is as in the figure on the right: red on the left (except for the bottom-left corner), yellow on the bottom (except for the bottom-right corner), and blue everywhere else. We will call this boundary coloring the *standard boundary coloring* and we will call a Sperner coloring satisfying this a *standard Sperner coloring*.



Assuming a standard Sperner coloring is without loss of generality. Indeed, if a given Sperner coloring is not standard (as in Figure 1), we can always augment the grid with an additional layer of boundary that is colored in the standard way, and embed the given Sperner coloring in the inside. Due to the properties of Sperner colorings and the standard boundary coloring, this operation will not introduce any trichromatic triangles between the extra boundary and the old boundary.

Now that our boundary coloring is standard, we can easily show Sperner's lemma using a graph-theoretic argument. Given a standard Sperner coloring we can define a directed graph, called *Sperner graph*, as follows:



- there are as many nodes in the graph as there are triangular cells in the grid; each cell of the grid is identified with a node of the graph;
- there is a directed edge $u \rightarrow v$ from node u to node v in the graph if their corresponding cells u and v in the grid share a *red-yellow* edge and, in order to go from cell u to cell v , one would have to cross this edge having the red color on the left and the yellow color on the right.

For the standard Sperner coloring of Figure 1, the corresponding graph is shown just above on the left.

L2.2.1 Properties of the Sperner graph

As you might have guessed from the picture, the following key properties hold.

Theorem L2.7. In any Sperner graph, the following properties hold:

- (1) every node has outdegree and indegree at most 1;
- (2) any node with indegree 1 and outdegree 0 is a trichromatic triangle (marked green in the figure above);
- (3) any node with outdegree 1 and indegree 0 is a trichromatic triangle (marked green), with the only exception of the bottom-left node (marked purple).

Proof.

- (1) follows by noticing that every cell has at most one *red-yellow* edge that one can use to exit this cell while keeping red on the left, and at most one *red-yellow* edge that one can use to enter this cell while keeping red on the left.
- (2) can be shown by contradiction. Take any node with indegree 1 and outdegree 0, and assume for contradiction that it is not a trichromatic triangle. Since the indegree is 1, one of the sides of the cell corresponding to the node is red-yellow and this edge can be crossed to enter into this cell from a neighboring cell. Let us now consider

the third vertex of the cell. Since by assumption the cell is not trichromatic, the third vertex is either red or yellow. Either case results in another red-yellow edge that one would be able to cross to exit the cell keeping red on the left. The only reason why this would not mean that the outdegree of the node corresponding to that cell is 1 is that this edge lies on the boundary of the grid. However, there are no such red-yellow edges on the boundary of the grid in the standard Sperner coloring. There is a unique red-yellow edge in the standard boundary coloring (at the bottom left cell) but this is an entry door, not an exit one.

(3) can be shown with a similar argument as (2).

□

■ L2.2.2 Completing the proof of Sperner's lemma

At this point, the proof of Sperner's lemma is immediate. A graph in which each node has indegree at most one and outdegree at most one is composed of connected components that can only be singleton nodes, directed paths, or directed simple cycles. Only paths have nodes with outdegree 1 and indegree 0, or outdegree 0 and indegree 1; each has exactly one of each. Note also that the standard boundary coloring forces the bottom left cell not to be trichromatic, and node corresponding to this cell to have outdegree 1 and indegree 0. So this node must be the source of a path. The sink of that path is trichromatic as per Theorem L2.7. If there are other paths, both their source and their sink are trichromatic, as per Theorem L2.7. Hence, there are an odd number of trichromatic triangles in any standard Sperner coloring, and therefore any Sperner coloring.

■ L2.3 Beyond the unit square

We stated and proved Sperner's lemma for the two-dimensional grid, and used that to prove Brouwer's fixed point theorem for continuous functions mapping the unit square to itself. There is a d -dimensional generalization of Sperner's lemma, which can be used to prove Brouwer's fixed point theorem for continuous functions mapping $[0, 1]^d$ to itself. In the high-dimensional case, a d -dimensional grid is partitioned into simplices, the d -dimensional analog of triangles, without introducing any more vertices other than those in the grid. The vertices of the grid are now colored with $d + 1$ colors, $0, 1, \dots, d$. Now, a coloring is valid if color i is not present in facet $x_i = 0$, for all $i = 1, \dots, d$, and color 0 is not present in all facets $x_i = 1$, for all $i = 1, \dots, d$. Sperner's lemma guarantees the existence of a simplex that has all $d + 1$ colors on its $d + 1$ vertices. Using the d -dimensional version of Sperner's lemma to prove Brouwer's fixed point theorem for continuous functions mapping the d -dimensional hypercube to itself is analogous to the $d = 2$ case. Finally, given Brouwer's fixed point theorem for the hypercube it is not hard to prove it for other convex and compact sets. Given a function defined on an arbitrary convex and compact set, one can first affinely transform the coordinate system so the set lies inside the unit hypercube. Then the function can be extended outside of the set by first projecting points of the hypercube to the set and then applying the function. This will not introduce any spurious fixed points.

Changelog

- Sep 24, 2025: fixed two typos (thanks Eric Yang Yu!)