18.06 Professor Strang Quiz 3 Solutions May 5, 1999

1 (a) $\lambda_1 = 1$ (because A is a Markov matrix) and $\lambda_2 = .3$ (from the trace).

$$S^{-1}AS = \begin{bmatrix} 4 & 1 \\ 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} .7 & .4 \\ .3 & .6 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & .3 \end{bmatrix}$$

(b) $A^k = S \wedge^k S^{-1} = \begin{bmatrix} 4 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & (.3)^k \end{bmatrix} \frac{1}{7} \begin{bmatrix} 1 & .1 \\ .3 & -4 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 4+3(.3)^k & 4-4(.3)^k \\ 3-3(.3)^k & 3+4(.3)^k \end{bmatrix}$

(c) The limit is $\frac{1}{7} \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \frac{2}{7} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, a multiple of the eigenvector x_1.

2 (a) $(2A - I)^{-1}$ has eigenvalues $\frac{1}{7}, \frac{1}{7}, -1$ so the determinant is $-\frac{1}{49}$.

(b) **True** (A is given as symmetric)

(c) **True** (no negative eigenvalues; A is positive semidefinite)

(d) **Not enough information**: $\frac{1}{4}A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ is not Markov, $\frac{1}{4}A = \begin{bmatrix} 1 & & \\ & .5 & .5 \\ & .5 & .5 \end{bmatrix}$ is Markov.

(e) $A = Q \wedge Q^T = \begin{bmatrix} & & \\ q_1 & q_2 & q_3 \\ & & \end{bmatrix} \begin{bmatrix} 4 & & \\ & 4 & \\ & & 0 \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ q_3^T \end{bmatrix} = 4q_1q_1^T + 4q_2q_2^T.$

3 (a) $x^TA^TAx = (Ax)^T(Ax)$ is positive unless $Ax = 0$. (Then it is zero — so if x is in the nullspace of A we do have $x^TA^TAx = 0$.)

(b) $A(A^TA)A^{-1} = (AA^T)$ so the matrices in parentheses are similar.

(c) $A^TA = (V\Sigma^TU^T)(U\Sigma V^T) = V\Sigma^T\Sigma V^T$.

4 (a) A has eigenvalues 1 and -1 (then $a+1$ and $a-1$).
R has eigenvalues i and $-i$ (then $b+i$ and $b-i$).

(b) We need $a+1 < 0$ for stability so the condition is a $a < -1$.

(c) We need the real parts of $b+i$ and $b-i$ to be negative, so the condition is $b < 0$.

(d) We need $a+1$ and $a-1$ to be positive, so the condition is $a > 1$.