18.06 Professor Strang Quiz 2 Solutions April 12, 1999

1 (a) $|A| = 2$ times the 3 by 3 determinant $= 2$ times $0 = 0$.

(b) A has rank 3 so we want three orthogonal basis vectors A, B, C:

$$\begin{aligned}
A &= \text{first column } (1,1,1,0) \\
B &= (\text{second column}) - (\text{projection onto first column}) \\
&= (-1,1,3,0) - (1,1,1,0)\cdot\frac{3}{3} \\
&= (-2,0,2,0) \quad \text{check: orthogonal to first column} \\
C &= \text{last column } (0,0,0,2)
\end{aligned}$$

To orthogonalize divide by lengths:

$$\begin{aligned}
q_1 &= \frac{A}{\|A\}} = \frac{A}{\sqrt{3}} \\
&= (1,1,1,0)/\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
q_2 &= \frac{B}{2\sqrt{2}} \\
&= (-1,0,1,0)/\sqrt{2}
\end{aligned}$$

$$\begin{aligned}
q_3 &= \frac{C}{2} \\
&= (0,0,0,1)
\end{aligned}$$

(c) Adding 1 to the a_{11} entry will add its cofactor to the determinant:

$$\text{Cofactor } C_{11} = \begin{vmatrix} 1 & 5 & 0 \\ 3 & 9 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -12\,.$$

2 (a)

$$\begin{bmatrix} 1 & -4 \\ 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

(b)

$$A^T A = \begin{bmatrix} 9 & 0 \\ 0 & 60 \end{bmatrix} \qquad A^T b = \begin{bmatrix} 10 \\ 40 \end{bmatrix}$$

Solve $A^T A \begin{bmatrix} C \\ D \end{bmatrix} = A^b$ to find $C = \frac{10}{9}$, $D = \frac{40}{60}$.

(c) The columns of A are a basis for the subspace. The projection is

$$p = C \text{ (column 1)} + D \text{ (column 2)}.$$

3 (a) Q has rank n (the n orthonormal) columns are independent).

(b) $P = Q(Q^T Q)^{-1} Q^T = QQ^T$.

(c) Check $P^T = P$: $(QQ^T)^T = QQ^T$.
Check $P^2 = P$: $(Q^T Q)Q^T = QQ^T$.

4 (a) The length of λ_x is $|\lambda| \|x\|$.
The length squared of Q_x is $(Qx)^T(Qx) = x^T Q^T Q x = x^T x = x^T x$.
Thus $|\lambda| \|x\| = \|x\|$ and $|\lambda| = 1$.
Note: We did not use the correct notation when λ and x are complex. The reasoning stays the same.

(b) Projection onto the last column:

$$p = a\frac{a^T b}{a^T a} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} 0 = \text{zero vector.}$$

Projection onto column space (which is all of R^4) is b itself.

(c) $|H_2 - \lambda I| = \begin{vmatrix} 1 - \lambda & 1 \\ 1 & -1 - \lambda \end{vmatrix} = \lambda^2 - 2 = 0.$
The eigenvalues are $\sqrt{2}$ and $-\sqrt{2}$. Check trace $= 0$ and determinant $= -2$.