Final Examination in Linear Algebra: 18.06

May 18, 1999 Solutions Professor Strang

1. (a) $\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

   (c) $5(\text{row } 1) + 4(\text{row } 2)$

   (d) A has rank 2 and A^T is 4 by 3 so its nullspace has dimension $3 - 2 = 1$.

2. (a) $C(A) = \mathbf{R}^5$ since every b is in the column space.

   (b) The rank is 5 so the five rows must be linearly independent.

   (c) The nullspace must have dimension $7 - 5 = 2$.

   (d) This is **false** because the 7 columns cannot be linearly independent.

3. (a) This is generally **false**, as for $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.
   Note that $A = LDU$ gives $A^{-1} = U^{-1}D^{-1}L^{-1}$ (upper times lower!).

   (b) **True** because $\det A^{-1} = 1/(\det A)$.

   (c) Multiply row 1 by A^{-1} and add to row 2 to obtain $\begin{bmatrix} A & I \\ 0 & A^{-1} \end{bmatrix}$.

   (d) The determinant is $+1$. Exchange the first n columns with the last n. This produces a factor $(-1)^n$ and leaves $\begin{bmatrix} I & A \\ 0 & -I \end{bmatrix}$ which is triangular with determinant $(-1)^n$. Then $(-1)^n(-1)^n = +1$.

4. (a) From $Ax_3 = \lambda_3 x_3$ we have $A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(b) $A = S \wedge S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & & \\ & 1 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$.

(c) Transpose $S^{-1}AS = \wedge$ to get $S^T A^T (S^{-1})^T = \wedge$. Then the columns of $(S^{-1})^T$ are the eigenvectors of A^T, and part (b) gives $(S^{-1})^T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

5. (a) $\begin{array}{l} 1C + 0D + E = 1 \\ 1C + 2D + E = 3 \\ 0C + 1D + E = 5 \\ 0C + 2D + E = 0 \end{array}$ is $Ax = b$.

(b) Subtract equation (1) from equation (2):

$$\begin{array}{rl} 2D \quad\;\; = 2 & \text{gives } D = 1 \\ D + E = 5 & \text{gives } E = 4 \\ 2D + E = 0 & \text{is now false} \end{array}$$

(c) Solve $A^T A\hat{x} = A^T b$:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \\ \hat{E} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 9 & 5 \\ 2 & 5 & 4 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \\ \hat{E} \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 7 & 3 \\ 0 & 0 & \frac{5}{7} \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \\ \hat{E} \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 2 \end{bmatrix}$$

Back-substitution gives $\hat{E} = \frac{14}{5}$, $\hat{D} = \frac{-1}{5}$, $\hat{C} = \frac{-3}{5}$.

(d) The error vector e is perpendicular to the three columns of A.

6. (a) One way is to solve for x perpendicular to q_1 and q_2:

$$\begin{bmatrix} 3 & 4 & 5 \\ 4 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Another way is Gram-Schmidt and we might as well start with $a_3 = (0, 0, 1)$. Then Gram-Schmidt subtracts off projections:

$$a_3 - (a_3^T q_1) q_1 - (a_3^T q_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{5}{50} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} - 0 = \begin{bmatrix} -.3 \\ -.4 \\ .5 \end{bmatrix}.$$

Normalizing to a unit vector gives

$$q_3 = \frac{1}{\sqrt{50}} \begin{bmatrix} -3 \\ -4 \\ 5 \end{bmatrix}.$$

(b) a_3 will not work if it is in the plane of q_1 and q_2.
The only possible vectors q_3 are $+(\text{our}\, q_3)$ and $-(\text{our}\, q_3)$.

(c) The projection is the vector that was subtracted off in part (a):

$$p = \frac{5}{50} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.5 \end{bmatrix}.$$

7. (a) Cannot exist because A and A^T have the same rank.

(b) $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or any non-square A with independent columns.

(c) The desired A has an eigenvalue like -2, outside the unit circle and in the left half-plane. In fact, $A = [-2]$ is a 1 by 1 example.

(d) From the two given nullspace vectors we know that $A = \begin{bmatrix} v & v & -v \end{bmatrix}$ for some column v. The particular solution $(1, 1, 1)$ determines v:

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{gives} \quad v + v - v = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{so} \quad v = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

(e) (My favorite this year)

The first pivot must be $a_{11} = -1$. The the trace $1 + 2$ requires $a_{22} = 4$. Then the determinant must be 2, so these matrices will work:

$$A = \begin{bmatrix} -1 & -1 \\ 6 & 4 \end{bmatrix} \quad \text{or any} \quad A = \begin{bmatrix} -1 & -a \\ 6/a & 4 \end{bmatrix}.$$

8. (a) $5! = 120$ terms are sure to be zero.

   (b) **Yes**, $(UV)^T(UV) = V^TU^TUV = V^TV = I$.

   (c) **No**, symmetry would need $AB = (AB)^T = B^TA^T = BA$ and we don't normally have $AB = BA$.

   (d) The 1 by 1, 2 by 2, 3 by 3 determinants are 1, $c-4$, and -4 (not depending on c!). The last is negative so A is not positive definite. But $\det A = -4$ so A has no zero eigenvalues so A^2 has all three positive eigenvalues.

9. (a) $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has $Ax_0 = 0$.

   (b) $A^2x_0 = A(Ax_0) = 0$

   (c) The dimension of $N(A^T)$ is at least 1 (because A is square and we know that $(1,1,1)$ is in $N(A)$).

   (d) A is singular so $\lambda = 0$ is an eigenvalue of A so $\lambda = 4$ is an eigenvalue of $A + 4I$.