Final Examination in Linear Algebra: 18.06

May 18, 1999 1:30 – 4:30 Professor Strang

Your name is: ______________________________

Secret Code (optional): ______________________

Please circle your recitation:

1)	Mon	2–3	2-131	S. Kleiman
2)	Mon	3–4	2-131	S. Hollander
3)	Tues	11–12	2-132	S. Howson
4)	Tues	12–1	2-132	S. Howson
5)	Tues	12–1	2-131	S. Kleiman
6)	Tues	1–2	2-131	S. Kleiman
7)	Tues	2–3	2-132	S. Howson

Grading
1
2
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9

Answer all 9 questions on these pages. This is a closed book exam. Calculators are not needed in any way and therefore not allowed (to be fair to all). *Grades are known only to your recitation instructor* (who is free to post with secret codes and to return exams in person). Solutions will be posted on the web in a few days. Best wishes for the summer and thank you for taking 18.06.

GS

1 Suppose the matrix A is the product

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) **(3 points)** Give a basis for the nullspace of A.

(b) **(3 points)** Give a basis for the row space of A.

(c) **(2 points)** Express row 3 of A as a combination of your basis vectors in your answer to (b).

(d) **(3 points)** What is the dimension of the nullspace of A^T?

2 Suppose A is a 5 by 7 matrix, and $Ax = b$ has a solution for every right side b.

(a) **(3 points)** What do we know about the column space of A?

(b) **(3 points)** What do we know about the rows of A?

(c) **(3 points)** What do we know about the nullspace of A?

(d) **(3 points)** True or false (with reason):

The columns of A are a basis for the column space of A.

Your answers could refer to dimension/basis/linear independence/spanning a space.

3 Assume that A is invertible and permutations are not needed in elimination if possible.

(a) **(2 points)** Are the pivots of A^{-1} equal to $\dfrac{\mathbf{1}}{\text{pivots of } A}$?

If yes, give a reason; if no, give an example.

(b) **(3 points)** Is the product of pivots of A^{-1} equal to $\dfrac{1}{\text{product of pivots of } A}$?

If yes, give a reason; if no, give an example.

(c) **(3 points)** Apply block elimination to the $2n$ by $2n$ matrix

$$M = \begin{bmatrix} A & I \\ -I & 0 \end{bmatrix}.$$

Multiply block row 1 by a suitable matrix and add to block row 2.

What matrix appears in the $(2,2)$ block?

(d) **(3 points)** What is the determinant of M? Explain the final plus or minus sign.

4 Suppose A has eigenvalues $\lambda_1 = 3$, $\lambda_2 = 1$, $\lambda_3 = 0$ with corresponding eigenvectors

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) **(3 points)** How do you know that the third column of A contains all zeros?

(b) **(3 points)** Find the matrix A.

(c) **(3 points)** By transposing $S^{-1}AS = \wedge$, find the eigenvectors y_1, y_2, y_3 of A^T.
(I am looking for specific vectors like x_1, x_2, x_3 above.)

5 This problem computes the plane $z = Cx + Dy + E$ that is closest (in the least squares sense) to these four measurements:

$$
\begin{array}{llll}
\text{At } x = 1, & y = 0 & \text{measurement gives} & z = 1 \\
\text{At } x = 1, & y = 2 & \text{measurement gives} & z = 3 \\
\text{At } x = 0, & y = 1 & \text{measurement gives} & z = 5 \\
\text{At } x = 0, & y = 2 & \text{measurement gives} & z = 0
\end{array}
$$

(a) **(3 points)** Write down the linear system $Ax = b$ with unknown vector $x = (C, D, E)$ that would give a plane going exactly through the four given points — except that this particular system has no solution.

(b) **(3 points)** Show that this system $Ax = b$ has no solution!

(c) **(3 points)** Find the best least squares solution $\widehat{x} = (\widehat{C}, \widehat{D}, \widehat{E})$.

(d) **(3 points)** The error vector $e = (e_1, e_2, e_3, e_4)$ in the underlying projection problem is perpendicular to which vectors? You don't have to compute e but you do have to say which specific numerical vectors it is perpendicular to.

6 (a) **(3 points)** The vectors $q_1 = \frac{1}{\sqrt{50}} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ and $q_2 = \frac{1}{5} \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$ are orthonormal.

Find one more vector to complete an orthonormal basis for $\mathbf{R}^3$.

(b) **(3 points)** In solving part (a), you might start with a vector like $a_3 = (0, 0, 1)$ and find q_3. Which vectors a_3 *would not work* as starting vectors to find q_3 by Gram-Schmidt? How many different real vectors q_3 will give a correct answer to part (a)?

(c) **(3 points)** Project the vector $a_3 = (0, 0, 1)$ onto the plane spanned by q_1 and q_2. Find its projection p.

7 In each part, find the required matrix or explain why such a matrix does not exist.

(a) **(3 points)** The matrices A and A^T and $A + A^T$ have ranks 2 and 1 and 3.

(b) **(3 points)** The solution to $Ax = 0$ is unique, but the solution to $A^Tx = 0$ is not unique.

(c) **(3 points)** The powers A^k do not approach the zero matrix as $k \to \infty$, but the exponential e^{At} does approach the zero matrix as $t \to \infty$.

(d) **(3 points)** The complete solution to

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{is} \quad x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

(e) **(3 points)** The pivots are -1 and -2 but the eigenvalues are $+1$ and $+2$. (Symmetric matrix not required, row exchanges not required.)

8 (a) **(3 points)** The "big formula" for a 6 by 6 determinant has $6! = 720$ terms. How many of those terms are sure to be zero if we know that $a_{15} = 0$?

(b) **(2 points)** If U and V are 3 by 3 orthogonal matrices, is their product UV always orthogonal? Why (give reason) or why not (give example)?

(c) **(2 points)** If A and B are 3 by 3 symmetric matrices, is their product AB always symmetric? Why (give reason) or why not (give example)?

(d) **(3 points)** For which numbers c is the matrix A positive definite? For which numbers c is A^2 positive definite? Why?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & c & 4 \\ 3 & 4 & 9 \end{bmatrix}$$

9 Suppose the 3 by 3 matrix A has the following property Z: Along each of its rows, the entries add up to zero.

(a) **(3 points)** Find a nonzero vector in the nullspace of A.

(b) **(3 points)** Prove that A^2 also has property Z.

(c) **(3 points)** What can you say about the dimension of the nullspace of A^T and why?

(d) **(2 points)** Find an eigenvalue of the matrix $A + 4I$.