18.06 Solutions Quiz 3 May 9, 1997

1.(a) A is a Markov matrix. So $\lambda = 1$ is an eigenvalue. Then $\lambda = .2$ is the other eigenvalue because trace $= 1.2$

Eigenvectors

$$\lambda = 1: \quad A - I = \begin{bmatrix} -.3 & .5 \\ .3 & -.5 \end{bmatrix} \rightarrow x = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\lambda = .2: \quad A - .2I = \begin{bmatrix} .5 & .5 \\ .3 & .3 \end{bmatrix} \rightarrow x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$S = \begin{bmatrix} 5 & 1 \\ 3 & -1 \end{bmatrix}$ has inverse $\frac{1}{8}\begin{bmatrix} 1 & 1 \\ 3 & -5 \end{bmatrix} = S^{-1}$ so that $S^{-1}u_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\frac{1}{8}$

We want

$$A^{20}u_0 = S \wedge^{20} S^{-1}u_0 = \begin{bmatrix} 5 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & \\ & .2^{20} \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}\frac{1}{8}$$

$$= \begin{bmatrix} 5 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 3x \quad .2^{20} \end{bmatrix}\frac{1}{8}$$

$$= \begin{bmatrix} 5 + 3 \quad (.2^{20}) \\ 3 - 3 \quad (.2^{20}) \end{bmatrix}\frac{1}{8}$$

Check: Change 20th power to 0th and we get $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = u_0$

(b) A has $\lambda = .2$ and 1

$A - cI$ has $\lambda = .2 - c$ and $1 - c$

Need $|.2 - c| < 1$ (need $c < 1.2$) and $|1 - c| < 1$ (need $c > 0$)

Therefore the condition for $|\lambda| < 1$ and stability is $0 < c < 1.2$

(c) Same eigenvectors for $A^{-1} + A^{20}$ as in part (a) for A itself

$$\text{Eigenvalues} \qquad \frac{1}{1} + 1^{20} = 2$$

$$\frac{1}{.2} + (.2)^{20}$$

2.(a) If you transpose $S^{-1}AS = \Lambda$ you learn that $S^TA^T(S^{-1})^T = \Lambda^T = \Lambda$

The eigenvalues of A^T are **the same as the eigenvalues of** A

The eigenvectors of A^T are **the columns of** $(S^{-1})^T$

(b)

$$B = \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{matrix}$$

for symmetry   to make B singular

(c) Multiply on the right by $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So subtract the 2nd column of K from the first.

Then the result $\begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{bmatrix} C \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{bmatrix}^{-1}$ is **similar** to C and has the same eigenvalues

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3.(a) (8pts) Check determinants

$$\begin{array}{rcll} 1 & > & 0 & 1 \times 1 \\ c-1 & > & 0 & 2 \times 2 \\ c-2 & > & 0 & 3 \times 3 \end{array}$$

$$\boxed{\text{Need } c > 2 \text{ for positive definiteness}}$$

(2 pts)

$$\begin{aligned} F &= \frac{1}{2}\left(x_1^2 - 2x_1x_2 + cx_2^2 - 2x_2x_3 + x_3^2\right) \\ &= \frac{1}{2}x^T A\, x\,. \end{aligned}$$

Then $\dfrac{\partial^2 F}{\partial x_i \partial x_j} = A_{ij} =$ second derivative matrix.

(b)

$$P = \frac{aa^T}{a^Ta} = \frac{\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} [\cos\theta \ \sin\theta]}{\cos^2\theta + \sin^2\theta} = \begin{bmatrix} \cos^2\theta & \cos\theta\ \sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \quad \boxed{\lambda = 1, 0}$$

(c) Since $Pv_1 = v_1$ and $Pv_2 = 0$ the projection matrix in this basis is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.