18.06 **Solutions** **Quiz 2** **April 4, 1997**

1. (a) The left nullspace (The nullspace of A^T)

(b)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 4 & 4 \\ 1 & 4 & 9 & 9 \\ 1 & 4 & 9 & 16 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & 3 & 8 & 8 \\ 0 & 3 & 8 & 15 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

$$\text{determinant} = 1 \cdot 3 \cdot 5 \cdot 7 = \boxed{105}$$

(c) The determinant is $\pm$ product of the pivots .

The sign is $(-1)^{\text{number of row exchanges}}$

Reason: Row exchanges reverse sign
Subtracting multiples of row i from j does not change determinant

Det of triangular matrix U = product of pivots on diagonal.

2. (a)

$$\begin{aligned} C + D &= b_1 \\ C + 2D &= b_2 \\ C + 3D &= b_3 \end{aligned} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

(b) Elimination gives (by subtracting equation 1):

$$\begin{aligned} D &= b_2 - b_1 \\ 2D &= b_3 - b_1 \end{aligned} \qquad \text{then} \qquad \begin{aligned} 0 &= (b_3 - b_1) - 2(b_2 - b_1) \\ &\boxed{0 = b_3 - 2b_2 + b_1} \end{aligned}$$

Other method:

A basis for the left nullspace of A is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ since $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Then by Question 1(a), b should be orthogonal to this vector which means $b_1 - 2b_2 + b_3 = 0$.

(c)

$$A^T A = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \qquad A^T b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\text{Solve} \quad \begin{array}{r} 3\bar{C} + 6\bar{D} = 1 \\ 6\bar{C} + 14\bar{D} = 3 \end{array} \quad \rightarrow \quad \begin{array}{c} 2\bar{D} = 1 \\ \boxed{\bar{D} = \tfrac{1}{2} \quad \bar{C} = -\tfrac{2}{3}} \end{array}$$

(d)

$$\begin{aligned} P &= A(A^T A)^{-1} A^T \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \frac{\begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}}{6} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 8 & 2 & -4 \\ -3 & 0 & 3 \end{bmatrix} \\ & \boxed{= \tfrac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}} \qquad \text{check} \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \text{is still in the nullspace} \end{aligned}$$

3. (a) The vectors v, q_1, q_2, q_3 must be linearly independent

(b)

$$q_4 = \frac{v - (v^T q_1)q_1 - (v^T q_2)q_2 - (v^T q_3)q_3}{\| \, v - (v^T q)q_1 - (v^T q_2)q_2 - (v^T q_3)q_3 \, \|}$$

Always OK to write $q^T v$ instead of $v^T q$ (for real vectors)

(c)

$$\begin{aligned} Aq_1 &= q_1 \underset{\displaystyle 1}{\underset{\downarrow}{q_1^T q_1}} + q_2 \underset{\displaystyle 0}{\underset{\downarrow}{q_2^T q_1}} + \cdots + q_n \underset{\displaystyle 0}{\underset{\downarrow}{q_n^T q_1}} \\ &= q_1 \end{aligned}$$

Similarly $Aq_i = q_i$. Then $\boxed{A = I}$ (since q's are a basis for $\mathbf{R}^n$).

Other method: (columns $\times$ rows)

$$A = [q_1 \ \dots \ q_n] \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} = QQ^T = QQ^{-1} = I$$