18.06 Problem Set 8 - Solutions

Due Wednesday, April 25, 2007 at **4:00 p.m.** in 2-106

Problem 1 *Wednesday 4/18*

Do problem 5 of section 6.3 in your book.

Solution 1

To show $v+w$ is constant we need to show $\frac{d}{dt}(v+w) = 0$. We have

$$\frac{d}{dt}(v+w) = \frac{dv}{dt} + \frac{dw}{dt} = w - v + v - w = 0$$

So $v+w$ is constant and since $v(0) = 30$ and $w(0) = 10$, it is equal to 40.
The matrix is $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. To find the eigenvalues we solve $\lambda^2 + 2\lambda = 0$. We get $\lambda_1 = 0$ and $\lambda_2 = -2$. The corresponding eigenvectors are $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
At $t = 1$ we have $v(1) = 20 + 10e^{-2}$ and $w(1) = 20 - 10e^{-2}$.

Problem 2 *Wednesday 4/18*

Do problem 11 of section 6.3 in your book.

Solution 2

(a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -i \end{bmatrix}$.
(b) $\mathbf{u}(t) = \frac{1}{2}e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \frac{1}{2}[e^{it} + e^{-it}; i(e^{it} - e^{-it})] = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$.

Problem 3 *Wednesday 4/18*

Let

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) What are the eigenvalues of A?
(b) How many linearly independent eigenvectors does A have? Find them.
(c) Find e^{At}.
(d) Find the solution to the differential equation $\frac{du}{dt} = Au$ when $u(0) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.

Solution 3

(a) A is upper triangular, so the eigenvalues are the entries in the diagonal: 0, 0, 0, 0.

(b) A has rank 3, so there is only one linearly independent eigenvector: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(c) Since A is not diagonalizable, we can't use diagonalization for this. Instead, let's compute the powers of A:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A^2 = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, $e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 = \begin{bmatrix} 1 & t & 2t = \frac{1}{2}t^2 & 3t + 2t^2 + \frac{1}{6}t^3 \\ 0 & 1 & t & 2t + \frac{1}{2}t^2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(d) $\mathbf{u}(t) = e^{At}\mathbf{u}(0) = \begin{bmatrix} 1 + 6t + \frac{5}{2}t^2 + \frac{1}{6}t^3 \\ 1 + 3t + \frac{1}{2}t^2 \\ 1 + t \\ 1 \end{bmatrix}$.

Note that you could have solved this system by back substitution as well, by first solving u_4 and going up.

Problem 4 *Friday 4/20*

Do problem 9 of section 6.4 in your book.

Solution 4

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of the matrix. If the three eigenvalues are real there is nothing to prove. So let $\lambda_1 = a + bi$ with $b \neq 0$. Then we know that $\lambda_2 = a - bi$. Also, $\lambda_1 + \lambda_2 + \lambda_3 = 2a + \lambda_3 = Trace$, and the trace is real, so λ_3 has to be real.

Problem 5 *Friday 4/20*

Do problem 16 of section 6.4 in your book.

Solution 5

(a)$\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} \mathbf{y} \\ -\mathbf{z} \end{bmatrix} = \begin{bmatrix} -A\mathbf{z} \\ A^T\mathbf{y} \end{bmatrix} = \begin{bmatrix} -\lambda\mathbf{y} \\ \lambda\mathbf{z} \end{bmatrix} = -\lambda\begin{bmatrix} \mathbf{y} \\ -\mathbf{z} \end{bmatrix}$. Thus we know that $-\lambda$ is an eigenvalues with corresponding eigenvector $\begin{bmatrix} \mathbf{y} \\ -\mathbf{z} \end{bmatrix}$.

(b)$A^TA\mathbf{z} = A^T(\lambda\mathbf{y}) = \lambda A^T\mathbf{y} = \lambda^2\mathbf{z}$. Thus we have that λ^2 is an eigenvalues for A^TA with corresponding eigenvector $\mathbf{z}$.

(c) From part (b) we know that λ^2 is an eigenvalue for $A^TA = I^TI = I$, so $\lambda^2 = 1$, thus $\lambda = \pm 1$. From part (a) we know that if λ is an eigenvalue so is $-\lambda$, thus we conclude that the eigenvalues are $1, 1, -1, -1$. Notice that for $\lambda = 1$, we must have $\mathbf{y} = \mathbf{z}$ and for $\lambda = -1$ we must have $\mathbf{y} = -\mathbf{z}$. Four eigenvectors for B are:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

Problem 6 *Friday 4/20*

Do problem 18 of section 6.4 in your book.

Solution 6

The nullspace and the *row* space are always perpendicular. But for a symmetric matrix, row space = column space. So if y is an eigenvector for $\lambda \neq 0$ (in the column space), it must be perpendicular to the set of eigenvectors for $\lambda = 0$ (the nullspace). *(And perpendicular to the other eigenspaces $\lambda = \beta$ too — use $A - \beta I$ (also symmetric) instead, and the same argument.)*

Problem 7 *Friday 4/20*

Do problem 27 of section 6.4 in your book.

Solution 7

The other eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (eigenvalue $\lambda = 1 + 10^{-15}$), which makes an angle with the other eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ of only $45^o = \pi/4$! *Moral: eigenvectors are very sensitive to roundoff error.*

Problem 8 *Monday 4/23*

Do problem 4 of section 6.5 in your book.

Solution 8

You can do this by partial derivatives, or you can do this using positive definite.
Note that
$f(x, y) = x^2 + 4xy + 3y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.
Thus, this equation has a minimum at $(0, 0)$ if and only if the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ is positive definite. The upper left determinants are 1 and -1, thus the matrix is not positive definite, f doesn't have a minimum at $(0, 0)$.
We can also see this my expressing f as a difference of squares;

$$f(x, y) = (x + 2y)^2 - y^2$$

When $x = -2$ and $y = 1$, we get $f = -1$.

Problem 9 *Monday 4/23*

Do problem 19 of section 6.5 in your book.

Solution 9

Since A is symmetric, it has orthogonal eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Then $\mathbf{x} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$, so

$$\mathbf{x}^T A\mathbf{x} = (c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n)^T A(c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n) = (c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n)^T (c_1\lambda\mathbf{x}_1 + \cdots + c_n\lambda\mathbf{x}_n).$$

Since the eigenvectors are orthogonal, we have $\mathbf{x}_i^T\mathbf{x}_j = 0$ if $i \neq j$, so

$$\mathbf{x}^T A\mathbf{x} = c_1^2\lambda\mathbf{x}_1^T\mathbf{x}_1 + \cdots + c_n^2\lambda\mathbf{x}_n^T\mathbf{x}_n > 0$$

when $\mathbf{x} \neq 0$ since $c_i^2 > 0$, $\lambda_i > 0$ and $\mathbf{x}_i^T\mathbf{x}_i = ||\mathbf{x}_i||^2 > 0$.

Problem 10 *Monday 4/23*

Let A be any 3×3 symmetric matrix. Is it true that for large enough t, $A + tI$ is positive definite?

Solution 10

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of A. Since A is symmetric they are real. The eigenvalues of $A+tI$ are $\lambda_1 + t, \lambda_2 + t, \lambda_3 + t$. We just need to take t large enough such that $\lambda_1 + t, \lambda_2 + t, \lambda_3 + t$ are all positive, which can be done by taking t to be larger than the absolute value of the smallest of the λ's.