18.06 Problem Set 5 - Solutions

Due Wednesday, March 21, 2007 at **4:00 p.m.** in 2-106

Problem 1 *Wednesday 3/14*

Do problem 10 of section 4.3 in your book.

Solution 1

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 1 & 3 & 9 & 27 & 8 \\ 1 & 4 & 16 & 64 & 20 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 6 & 24 & -16 \\ 0 & 0 & 12 & 60 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 6 & 24 & -16 \\ 0 & 0 & 0 & 12 & 20 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 6 & 0 & -56 \\ 0 & 0 & 0 & 12 & 20 \end{bmatrix}$$

$F = 5/3, E = -28/3, D = 47/3, C = 0$

This system of linear equations is solvable, so p= $\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ and e= $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Problem 2 *Wednesday 3/14*

Do problem 17 of section 4.3 in your book.

Solution 2

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$.

$\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix}$ for the least square approximation is the one satisfying the following equation.

$$A^T A \begin{bmatrix} C \\ D \end{bmatrix} = A^T \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix} \rightarrow \hat{x} = \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

Problem 3 *Wednesday 3/14*

Find a function of the form $f(t) = Csin(t) + Dcos(t)$ that approximates the three points $(0, 0)$, $(\pi/2, 2)$, and $(\pi, 1)$. In other words, find coefficients C and D such that the error $|f(0) - 0|^2 + |f(\pi/2) - 2|^2 + |f(\pi) - 1|^2$ is as small as possible.

Solution 3

We minimize this expression using least square approximation. The system of equations we would like to solve is:

$$\begin{aligned} 0 \cdot C \quad &+1 \cdot D \quad &= 0 \\ 1 \cdot C \quad &+0 \cdot D \quad &= 2 \\ 0 \cdot C \quad &-1 \cdot D \quad &= 1 \end{aligned}$$

We get these equations by plugging $t = 0, \pi/2, \pi$. In matrix notation this is the same as

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

.
We multiply by A^T and get

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

so we get $C = 2$ and $D = -1/2$.

We could also solve this problem directly: $|f(0)-0| = D$, $|f(\pi/2)-2| = C-2$, $|f(\pi)-1| = -D-1$
$|f(0)-0|^2 + |f(\pi/2)-2|^2 + |f(\pi)-1|^2 = D^2 + (C-2)^2 + (-D-1)^2 = 2D^2 + 2D + 1 + (C-2)^2 = 2(D+1/2)^2 + (C-2)^2 + 1/2$
This is minimum when C=2 and D=-1/2

Problem 4 *Wednesday 3/14*

The `MATLAB` command `a=ones(n,1)` produces and n-by-1 matrix of ones. The command `r=(1:n)'` gives the vector $(1, 2, \ldots, n)$ transposed to a column by '. The command `s=r.^3` gives the column vector $(1^3, 2^3, \ldots, n^3)$, because the dots mens "a component at a time."
The purpose of this problem is to find the line $y = c + dt$ closest to the cubic function $y = t^3$ on the interval $t = 0$ to $t = 1$.
(a) Find the best line using calculus, not `MATLAB`. Choose c and d to minimize

$$E(c,d) := \int_0^1 (c + dt - t^3)^2 dt$$

(Hint: find $E(c,d)$ in terms of c and d, and use any method learned in 18.02 to minimize this.)

(b) With $n = 15$, choose C and D to give the line $y = C + Dt$ that is closest to t^3 at the points $t = \frac{1}{15}, \frac{2}{15} \ldots, 1$. Use `MATLAB` to do least squares in order to find C and D, and the differences $c - C$ and $d - D$.

(Hint: Set up the equations you want to solve. Your equations (in matrix form) should involve the vectors `a`, `r/n` and `s/n^3`).

(c) Repeat for $n = 30$. (Notice how `r/n` and `s/n^3` end at 1). Are the differences $c - C$ and $d - D$ smaller for $n = 30$? By what factor?

Solution 4

(a) $E(c,d) := \int_0^1 (c + dt - t^3)^2 dt = \int_0^1 t^6 + d^2t^2 + c^2 + 2cdt - 2ct^3 - 2dt^4 dt$
$= [\frac{1}{7}t^7 - \frac{1}{2}ct^4 - \frac{2}{5}dt^5 + \frac{1}{3}d^2t^3 + c^2t + cdt^2]_0^1$
$= \frac{1}{7} - \frac{c}{2} - \frac{2}{5}d + \frac{d^2}{3} + c^2 + cd = (c + d/2 - 1/4)^2 + (d - 10/9)^2/12 - 1/16 - 100/81 + 1/7$
This is minimum when d=9/10 and c=-1/5.

(b) D=1.0018, C=-0.2498; d-D=0.1018, c-C=0.0498

(c) D=0.9504, C=-0.2241; d-D=0.0504, c-C=0.0241

The difference is smaller for n=30 than for n=15 by about factor 1/2.

Problem 5 *Friday 3/16*

Consider in $\mathbb{R}^4$ the subspace given by $F = \{(x, y, z, w) : -x + y + 2z - w = 0\}$.

(a) Give a basis for F.

(b) Use Gram-Schmidt to transform your basis into an orthonormal basis.

(c) What is the distance between the point (1, 3, 1, 1) and (the closest point to) F?

Solution 5

(a) $w = -x+y+2z \rightarrow (x, y, z, w) = (x, y, z, -x+y+2z) = x(1, 0, 0, -1)+y(0, 1, 0, 1)+z(0, 0, 1, 2)$
Therefore, basis of F=$\{(1, 0, 0, -1), (0, 1, 0, 1), (0, 0, 1, 2)\}$

(b) Let $\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$.

$A = a$

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 1/2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} - (-2)/2 * \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - 4/6 * \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ 1 \\ 2/3 \end{bmatrix}$$

$\{A, B, C\}$ is orthogonal basis of F
After normalizing, We get orthonomal basis of F,

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{21}} \begin{bmatrix} 2 \\ -2 \\ 3 \\ 2 \end{bmatrix} \right\}.$$

(c) The closet point in F to the point (1,3,1,1) is

$$QQ^T \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 6 & 1 & 2 & -1 \\ 1 & 6 & -2 & 1 \\ 2 & -2 & 3 & 2 \\ -1 & 1 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 10 \\ 18 \\ 1 \\ 10 \end{bmatrix}.$$

Another way of computing the projection given an orthonormal basis, is as the sum of the projections along each basis vector:

$$0 * \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \frac{8}{\sqrt{6}} * \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{\sqrt{21}} * \frac{1}{\sqrt{21}} \begin{bmatrix} 2 \\ -2 \\ 3 \\ 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 10 \\ 18 \\ 1 \\ 10 \end{bmatrix}.$$

To compute the distance we compute $d^2 = (3/7)^2 + (3/7)^2 + (6/7)^2 + (3/7)^2 = 9/7$, so $d = 3/\sqrt{7}$.

Problem 6 *Friday 3/16*

Do problem 6 of section 4.4 in your book.

Solution 6

Q_1Q_2 is orthogonal iff $(Q_1Q_2)^T(Q_1Q_2) = I$.
$(Q_1Q_2)^T(Q_1Q_2) = Q_2^TQ_1^TQ_1Q_2 = Q_2^T(Q_1^TQ_1)Q_2 = Q_2^TQ_2 = I$.
Therefore, Q_1Q_2 is orthogonal.

Problem 7 *Friday 3/16*

(a)Find a 3-by-3 orthogonal matrix A such that
$A\begin{bmatrix}1\\0\\0\end{bmatrix} = 1/\sqrt{3} * \begin{bmatrix}1\\1\\1\end{bmatrix}$ and $A\begin{bmatrix}0\\1\\0\end{bmatrix} = 1/\sqrt{2} * \begin{bmatrix}1\\0\\-1\end{bmatrix}$
(b)How many matrices A are there that satisfies this conditions?

Solution 7

$A\begin{bmatrix}1\\0\\0\end{bmatrix} = 1/\sqrt{3} * \begin{bmatrix}1\\1\\1\end{bmatrix} \rightarrow$ first column of $A = 1/\sqrt{3} * \begin{bmatrix}1\\1\\1\end{bmatrix}$
$A\begin{bmatrix}0\\1\\0\end{bmatrix} = 1/\sqrt{2} * \begin{bmatrix}1\\0\\-1\end{bmatrix} \rightarrow$ second column of $A = 1/\sqrt{2} * \begin{bmatrix}1\\0\\-1\end{bmatrix}$
Therefore, A=$\begin{bmatrix}1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3}\\1/\sqrt{2} & 0 & -1/\sqrt{2}\\x & y & z\end{bmatrix}$
A is orthonormal $\rightarrow x^2 + y^2 + z^2 = 1$, $1/\sqrt{2} * (x - z) = 0$, $1/\sqrt{3} * (x + y + z) = 0$
$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x\\-2x\\x\end{bmatrix}$ and $6x^2 = 1$
$A = \begin{bmatrix}1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3}\\1/\sqrt{2} & 0 & -1/\sqrt{2}\\1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6}\end{bmatrix}$ or $A = \begin{bmatrix}1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3}\\1/\sqrt{2} & 0 & -1/\sqrt{2}\\-1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6}\end{bmatrix}$ ...(a)

Therefore, there's two matrices A that satisfies this condition. ...(b)

Problem 8 *Monday 3/19*

Do the problem 9 of section 5.1 in your book.

Solution 8

$$detA = 1 * det\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = 1$$

$$detB = -1 * det\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} + 1 * det\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} = 2$$

$$detC = det\begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} = 0$$

Problem 9 *Monday 3/19*

Calculate the determinant of the matrix $A = \begin{bmatrix}1 & 2 & 3 & 4 & 5\\2 & 3 & 4 & 5 & 6\\3 & 4 & 5 & 6 & 7\\4 & 5 & 6 & 7 & 8\\5 & 6 & 7 & 8 & 9\end{bmatrix}$

Solution 9

$detA = det\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \end{bmatrix}$, subtracting the first row from all the other rows.

This matrix is clearly singular, thus $detA = 0$.

Problem 10 *Monday 3/19*

(a)Calculate the determinants of the following "almost upper triangular" matrices.

$A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$A_3 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

$A_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

$A_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$

(b) Can you figure out how to continue the sequence of matrices and calculate $det(A_n)$ for any n?

Solution 10

(a)$detA_2 = 2$

$detA_3 = det\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} det\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & -1 & 1 \end{bmatrix} = det\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & -1 & 1 \end{bmatrix} = 2 * det\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = 2 * detA_2 = 4$

$detA_4 = det\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = det\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = 2 * det\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = 2 * detA_3 = 8$

$detA_5 = det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} = det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$

$= 2 * det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} = 2 * detA_4 = 16$

(b) $[A_n]_{ij} = 1$ if $i \leq j$, $[A_n]_{ij} = -1$ if $i = j + 1$, $[A_n]_{ij} = 0$ otherwise.

As above, $detA_n = 2 * A_{n-1} = 2^{n-1}$