18.06 Problem Set 4 - Solutions

Due Wednesday, Mar. 14, 2007 at **4:00 p.m.** in 2-106

Problem 1 *Wednesday 2/28*

Do problem 37 of section 3.5 in your book.

Solution 1

We want to find a basis for the space of polynomials of degree ≤ 3, i.e., the space that contains all polynomials of the form $p(x) = ax^3 + bx^2 + cx + d$, with $a, b, c, d \in \mathbb{R}$. A possible basis is $\left\{x^3, x^2, x, 1\right\}$ (just set each of the "free variables" a, b, c, d to 1 and all others to 0, and do this for each one). It is easy to see that these are linearly independent and span the space.
Now, we want to find a basis for the subspace of polynomials of degree ≤ 3 that satisfy $p(1) = 0$. First, note that for the general element of the space above, $p(x) = ax^3 + bx^2 + cx + d$, we have $p(1) = a + b + c + d$. Thus, $p(1) = 0$ if and only if $d = -(a + b + c)$. So what we want is a basis of the subspace of all polynomials of the form $p(x) = ax^3 + bx^2 + cx - (a + b + c)$, with $a, b, c, d \in \mathbb{R}$. A possible basis is $\left\{x^3 - 1, x^2 - 1, x - 1\right\}$, which again clearly spans the subspace and is a linearly independent set.

Problem 2 *Wednesday 2/28*

Do problem 28 of section 3.6 in your book (including the challenge problem). For the challenge problem, assume r, n, b, q, k, p are all nonzero.

Solution 2

Let B be the eight by eight checkerboard matrix. The first, third, fifth and seventh rows of $B$are all equal to $(1, 0, 1, 0, 1, 0, 1, 0)$, whereas the second, fourth, sixth and eighth rows are all equal to $(0, 1, 0, 1, 0, 1, 0, 1)$. Since these two row vectors are linearly independent, a basis for the row space of B is the set

$$\left\{(1, 0, 1, 0, 1, 0, 1, 0)^T, (0, 1, 0, 1, 0, 1, 0, 1)^T\right\}.$$

The rank of B is 2, equal to the dimension of the row space. The left nullspace of B is the space of all vectors y such that $B^T y = 0$. Note that $B^T = B$, so the left nullspace is the same as the nullspace. These vectors must satisfy $y_1 + y_3 + y_5 + y_7 = 0$ and $y_2 + y_4 + y_6 + y_8 = 0$. Rewriting, we have $y_7 = -(y_1 + y_3 + y_5)$ and $y_8 = -(y_2 + y_4 + y_6)$. So a basis for the left nullspace is the set consisting of the vectors $(1, 0, 0, 0, 0, 0, -1, 0)^T$, $(0, 0, 1, 0, 0, 0, -1, 0)^T$, $(0, 0, 0, 0, 1, 0, -1, 0)^T$, $(0, 1, 0, 0, 0, 0, 0, -1)^T$, $(0, 0, 0, 1, 0, 0, 0, -1)^T$ and $(0, 0, 0, 0, 0, 1, 0, -1)^T$.

Let C be the chess matrix the first and eighth rows are (r, n, b, q, k, b, n, r), the second and seventh are (p, p, p, p, p, p, p, p) and all others are zero rows. Since these two nonzero rows are linearly independent (r, n, b, q, k are distinct, so the first row cannot be a multiple of the second), a basis for the row space of C is the set

$$\left\{(r, n, b, q, k, b, n, r)^T, (p, p, p, p, p, p, p, p)^T\right\}.$$

If $p = 0$, the basis for the row space is $\left\{(r, n, b, q, k, b, n, r)^T\right\}$. The rank of C is 2 if $p \neq 0$ and 1 if $p = 0$. The left nullspace of C is the set spanned by the vectors $(1, 0, 0, 0, 0, 0, 0, -1)^T$, $(0, 1, 0, 0, 0, 0, -1, 0)^T$, $(0, 0, 1, 0, 0, 0, 0, 0)^T$, $(0, 0, 0, 1, 0, 0, 0, 0)^T$, $(0, 0, 0, 0, 1, 0, 0, 0)^T$, $(0, 0, 0, 0, 0, 1, 0, 0)^T$. If $p = 0$, the basis for the left nullspace of C is the same set as above but removing $(0, 1, 0, 0, 0, 0, -1, 0)^T$ and including $(0, 1, 0, 0, 0, 0, 0, 0)^T$, $(0, 0, 0, 0, 0, 0, 1, 0)^T$.

Challenge question:
For this we assume that r, n, b, q, k, p are all nonzero. We want to find a basis for the nullspace of C.

$$C = \begin{bmatrix} r & n & b & q & k & b & n & r \\ p & p & p & p & p & p & p & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p & p & p & p & p & p & p & p \\ r & n & b & q & k & b & n & r \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & \frac{n-b}{n-r} & \frac{n-q}{n-r} & \frac{n-k}{n-r} & \frac{n-b}{n-r} & 0 & 1 \\ 0 & 1 & \frac{b-r}{n-r} & \frac{q-r}{n-r} & \frac{k-r}{n-r} & \frac{b-r}{n-r} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for the nullspace of C consists of the vectors $(n-b, b-r, r-n, 0, 0, 0, 0, 0)^T$, $(n-q, q-r, 0, r-n, 0, 0, 0, 0)^T$,$(n-k, k-r, 0, 0-r-n, 0, 0, 0)^T$,$(n-b, b-r, 0, 0, 0, r-n, 0, 0)^T$, $(0, -1, 0, 0, 0, 0, 1, 0)^T$ and $(-1, 0, 0, 0, 0, 0, 0, 1)^T$.

Problem 3 *Wednesday 2/28*

Using `MATLAB`, take several random 4-by-4 matrices (try using the `rand(m,n)` function) and look at their four subspaces. (A convenient way to calculate the subspaces is the `fourbase.m` teaching code; type in `type fourbase` at the `MATLAB` prompt for information on how to use it.[1]) What are the dimensions of the four subspaces for a "typical" 4-by-4 matrix? Can you explain why? *(Hint: what are the odds a pivot is exactly zero?)*
Now try 4-by-2 matrices. What are the dimensions of the four subspaces now? Now guess what dimensions the four subspaces of a random m-by-n matrix will most likely have.

Solution 3

Here's one example of the 4-by-4 case:

```
>> [R,N,C,L] = fourbase(rand(4,4))

R =
    1.0000         0         0         0
         0    1.0000         0         0
         0         0    1.0000         0
         0         0         0    1.0000

N =
   Empty matrix: 4-by-0

C =
    0.8180    0.3412    0.8385    0.5466
    0.6602    0.5341    0.5681    0.4449
    0.3420    0.7271    0.3704    0.6946
    0.2897    0.3093    0.7027    0.6213

L =
   Empty matrix: 4-by-0
```

Unless we're very unlucky, we always get similar results: the row and column space have dimension 4, the nullspace and the left nullspace have dimension 0 (containing only the zero vector). *(In other words, almost all random matrices have full rank.)* The chance that there is a linear combination of the columns that sums to zero is negligible. Indeed, think of doing elimination on this random matrix. The probability that the (1,1) entry is exactly zero is negligible, so this is our first pivot and we may begin the elimination. Subtracting multiples of the first row to the second, third and fourth row, we get zero on all entries under the first pivot. The probability that the number now on position (2,2) is exactly zero is negligible (for that, we would have

[1] If you need to download the file `fourbase.m` from the Web site, don't forget to put it in the current directory where `MATLAB` can find it.

to have exactly $a_{22} = \frac{a_{21}}{a_{11}} a_{12}$ on the original matrix, which is very unlikely since the entries were picked at random). Thus, we have a second pivot, and by elimination we get zeros on the entries of the second column under the second pivot. The same reasoning continues until we finally have an upper triangular matrix, with as many pivots as possible (in all columns or in all rows), and this matrix has full rank.
Here's one example of the 4-by-2 case:

```
>> [R,N,C,L] = fourbase(rand(4,2))

R =
    1.0000         0
         0    1.0000

N =
   Empty matrix: 2-by-0

C =
    0.7948    0.1730
    0.9568    0.9797
    0.5226    0.2714
    0.8801    0.2523

L =
    1.0000         0
         0    1.0000
    0.4515   -5.8000
   -1.1711    2.3567
```

Again, as explained above, almost all random matrices have full rank. In this case, that is 2. That implies that the row and column space have dimension 2, the nullspace has dimension 0, and the left nullspace has dimension 2.

Problem 4 *Wednesday 3/7*

Do problems 11 and 12(a) of section 8.2 in your book.

Solution 4

11(a) The diagonal of $A^T A$ tells you how many edges in total go out and into each node.

11(b) The off-diagonals -1 or 0 tell which pairs of nodes are connected or not.

12(a)You learned in class that $N(A^T A) = N(A)$. Since the nullspace of A is the space generated by $(1, 1, 1, 1)^T$, so is the nullspace of $A^T A$. Since the rank of $A^T A$ plus the dimension of the nullspace of $A^T A$ equals n, we get $\text{rank}(A^T A) = n - 1$.
Another way of seeing that the vector $(1, 1, 1, 1)^T$ is in $N(A^T A)$ is the following: if you add all entries of row i of $A^T A$, you get $k - 1 - \ldots - 1$, where k is the diagonal element and the negative ones occur on some off-diagonal entries, the remaining of which are zero. But k is equal to the number of edges that begin or end at node i, and there are as many -1's as the number of nodes that are connected to node i, that is, k.
Thus, adding all entries of row i yelds 0, which will be the i-th entry of the vector $(A^T A) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Problem 5 *Friday 3/9*

Do problem 17 of section 4.1 in your book.

Solution 5

Let S be the subspace of $\mathbb{R}^3$ containing only the zero vector. Then $S^\perp$ is the space of all vectors $\mathbb{R}^3$ perpendicular to all vectors of S, that is, all vectors that dotted with 0 yeld zero. But this is true for all vectors in $\mathbb{R}^3$, so we have $S^\perp = \mathbb{R}^3$.
Now, let S be the vector subspace spanned by the vector $(1,1,1)$. Then $S^\perp$ is the subspace of all vectors such that $(x_1, x_2, x_3) \cdot (1,1,1) = x_1 + x_2 + x_3 = 0$. So, $S^\perp$ is the subspace spanned by the vectors $(-1,0,1)$ and $(-1,1,0)$.
Finally, let S be the vector subspace spanned by $(2,0,0)$ and $(0,0,3)$. Then $S^\perp$ is the subspace of all vectors such that $(x_1, x_2, x_3) \cdot (2,0,0) = 2x_1 = 0$ and $(x_1, x_2, x_3) \cdot (0,0,3) = 3x_3 = 0$, that is, such that $x_1 = x_3 = 0$. In other words, $S^\perp$ is spanned by $(0,1,0)$.

Problem 6 *Friday 3/9*

Do problem 25 of section 4.1 in your book.

Solution 6

Suppose that A is a matrix whose columns are the vectors $v_1, v_2, \ldots, v_n$, where these are unit vectors, mutually perpendicular. This means that $v_i \cdot v_j$ is equal to 1 if $i = j$ and equal to 0 if $i \neq j$. The ij-th entry of the matrix $A^T A$ is equal to $v_i \cdot v_j$, hence this is the diagonal matrix.

Problem 7 *Friday 3/9*

Do problem 29 of section 4.1 in your book.

Solution 7

The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$ contains the vector $v = (1,2,3)$ in its row space and column space. This implies that the v cannot be contained in the nullspace or in the left nullspace (because the row space is orthogonal to the nullspace, so if $v \neq 0$ is in one it cannot be in the other. The same happens with the column space and the left nullspace).
The matrix $B = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$ contains the vector $v = (1,2,3)$ in its nullspace and column space. This implies that v cannot be in the row space or the left nullspace.

Problem 8 *Monday 3/11*

Do problem 17 of section 4.2 in your book.

Solution 8

We have a projection matrix P such that $P^2 = P$. Then $(I-P)^2 = (I-P)(I-P) = I - P - P + P^2 = I - P - P + P = I - P$. This means that $(I - P)$ is itself a projection!
When P projects onto the column space of A, $I - P$ projects onto the orthogonal complement of $C(A)$, that is, onto the left nullspace of A. Indeed, recall that if $p = Pb$ is the projection of b onto a subspace, then the error $e = b - p = (I - P)b$ is in the orthogonal complement of that subspace.

Problem 9 *Monday 3/11*

Do problem 23 of section 4.2 in your book.

Solution 9

If the matrix A is square and invertible, then $P = A(A^TA)^{-1}A^T = AA^{-1}(A^T)^{-1}A^T = I$. In fact, if the matrix A is square and invertible, then its columns span the whole space $\mathbb{R}^n$, and thus projecting onto that space is the same as not doing anything, so $P = I$. The error $e = b - p$ will be zero.

Problem 10 *Monday 3/11*

We found in class an expression for the projection matrix P that projects a vector b onto the column space of a matrix A.
(a) Find a matrix M that projects a vector onto the left nullspace of A.
(b) What is the product PM? Explain your answer.

Solution 10

(a) From class we know that the matrix $P = A(A^TA)^{-1}A^T$ projects onto the column space of A. As seen in problem 8, $(I - P)$ projects onto the orthogonal complement of $C(A)$, that is, onto the left nullspace of A. Thus, $M = I - A(A^TA)^{-1}A^T$.
(b) $PM = 0$. We can see this in two ways: First, for any vector b, its projection Mb is contained in $C(A)^\perp$, and then projecting it onto $C(A)$ (the orthogonal complement of the space Mb is in) yelds zero, so $PMb = 0$. The second way is to write

$$PM = P(I - P) = P - P^2 = P - P = 0.$$