# 18.06 Problem Set 3 - Solutions

Due Wednesday, Feb. 28, 2007 at **4:00 p.m.** in 2-106

Each problem is worth 10 points. The date next to the problem number indicates the lecture in which the material is covered.

**Problem 1** *Tuesday 2/20*

For each of the following questions, please explain your answer.

(a) Let $F = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \geq y \geq z \geq 0\}$. Is $F$ a subspace of $\mathbb{R}^3$?

(b) The set of all real functions forms a vector space. Is the set of all functions of the form $f(x) = ax^2$ (where $a$ can take any real value) a subspace? How about the set of all functions of the form $f(x) = x^2 + bx + c$ (where $b$ and $c$ can take any real value)?

(c) Let $A$ be a fixed $3 \times 2$ matrix. Let $F$ be the set of all $3 \times 3$ matrices $B$ such that $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Is $F$ a subspace of the space of all $3 \times 3$ matrices?

***Solution 1***

(a) No, since if we multiply a vector from $F$ by a negative scalar, the result will not anymore be in $F$.

(b) For $ax^2$ the answer is yes, since for any two functions $f, g$ of the given form, $f(x) = ax^2$ and $g(x) = bx^2$ the sum $(f + g)(x) = (a + b)x^2$ and the result of multiplying $f$ by a scalar $c$ $cf(x) = (ca)x^2$ are still of the form scalar times $x^2$.

For $x^2 + bx + c$ the answer is no, since for two functions $f, g$ of the given form, $f(x) = x^2 + b_1x + c_1$ and $g(x) = x^2 + b_2x + c_2$ the sum $(f + g)(x) = 2x^2 + (b_1 + b_2)x + (c_1 + c_2)$ and here $x^2$ has coefficient 2 as opposed to 1.

(c) Yes, it is a subspace since if $B_1, B_2 \in F$, then we have $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = B_1A + B_2A = (B_1 + B_2)A$ and for any scalar $c$ if $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ then so is $cBA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

**Problem 2** *Tuesday 2/20*

Do problem 22 of section 3.1 in your book.

***Solution 2***

There are three systems in the problem. We can obviously solve the first system for any $(b_1, b_2, b_3)$. The second system can be solved provided $b_3 = 0$ since the last equation read $0x_1 + 0x_2 + 0x_3 = b_3$. Finally, in the third system we must have $b_2 = b_3$ since the second and third equations read $x_3 = b_2$ and $x_3 = b_3$.

**Problem 3** *Wednesday 2/21*

Do problems 5, 6 and 7 of section 3.2 in your book.

***Solution 3***

Problem 5. (a) $A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix}$. The corresponding $U = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) $B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}$. The corresponding $U = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.
Problem 6. The free variables corresponding to $A$ are $x_2$ and $x_3$. Therefore, we have two special solutions: $x_2 = 0$, $x_3 = 1$, $x_1 = 5$ and $x_2 = 1$, $x_3 = 0$, $x_1 = 3$.
There is only one free variable corresponding to $B$, namely $x_2$. Therefore, we have one special solution: $x_2 = 1$, $x_3 = 0$, $x_1 = 3$.
For an $m$ by $n$ matrix the number of pivot variables plus the number of free variables is $n$, the number of columns.
Problem 7.
$N(A) = \{c_1 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} : c_1, c_2 \text{ any scalars}\} = \{\begin{bmatrix} 5c_1 + 3c_2 \\ c_2 \\ c_1 \end{bmatrix} : c_1, c_2 \text{ any scalars}\}$
The equation describing $N(A)$ is $x = 3y + 5z$.
$N(B) = \{c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} : c_1 \text{ any scalar}\} = \{\begin{bmatrix} 3c_1 \\ c_1 \\ 0 \end{bmatrix} : c_1 \text{ any scalar}\}$
The equation describing $N(B)$ is $x = 3y$.

**Problem 4** *Wednesday 2/21*

Do problem 25 of section 3.2 in your book.

***Solution 4***

A possible answer is: $\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$. Since it has 3 pivots we know there is only one free variable, so the nullspace is exactly all the multiples of $(1, 1, 1, 1)$ and nothing more.

**Problem 5** *Friday 2/23*

Do problem 7 of section 3.4 in your book.

***Solution 5***

We have $A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 2 & 4 & 0 \end{bmatrix}$ and then if we augment it by the vector $(b_1, b_2, b_3)$ and do elimination we get the matrix $\begin{bmatrix} 1 & 3 & 1 & b_1 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + 4b_1 \end{bmatrix}$. Since the system is then solvable exactly when $b_3 - 2b_2 + 4b_1 = 0$ this implies that $(b_1, b_2, b_3)$ is in the column space exactly when $b_3 - 2b_2 + 4b_1 = 0$. From this equation we also read of that $r_3 - 2r_2 + 4r_1 = 0$, where $r_i$ denotes the $i$-th row.

**Problem 6** *Friday 2/23*

Do problem 21 of section 3.4 in your book.

***Solution 6***

(a) In the system $x + y + z = 4$ the free variables are $y$ and $z$, therefore the particular solution is $x_p = (4, 0, 0)$ and the two special solutions are $(-1, 1, 0)$ and $(-1, 0, 1)$. Thus the complete solution is of the form $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

(b) By Gaussian elimination the system reduces to $x+y+z=4$ and $-2y=0$ so the free variable is $z$. The particular solution is then $x_p = (4,0,0)$ and the special solution is $x_n = (-1,0,1)$ and the complete solution is $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

**Problem 7** *Friday 2/23*

Let $A=\begin{bmatrix} 1 & 2 & -2 & 1 & 0 \\ 2 & 4 & -3 & 3 & 0 \\ -3 & -6 & 5 & 4 & 2 \\ 5 & 10 & -9 & 6 & 0 \end{bmatrix}$.

(a) Transform $A$ to (ordinary) echelon form.
(b) What are the pivots? What are the free variables?
(c) Now transform $A$ to row reduced echelon form.
(d) Give the special solutions. What is the nullspace $N(A)$?
(e) What is the rank of $A$?
(f) Give the complete solution to $Ax=b$, where $b = A\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

***Solution 7***

(a) The echelon form is $\begin{bmatrix} 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 8 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(b) The pivots are the $(1,1)$, $(2,3)$ and $(3,4)$ entries, and so the pivot variables are $x_1$, $x_3$ and $x_4$ leaving $x_2$, $x_5$ as the free variables.

(c)We get $\begin{bmatrix} 1 & 2 & 0, 0, -0.75 \\ 0 & 0 & 1, 0, -0.25 \\ 0 & 0 & 0, 1, 0.25 \\ 0 & 0 & 0, 0, 0 \end{bmatrix}$.

(d) The special solutions are $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0.75 \\ 0 \\ 0.25 \\ -0.25 \\ 1 \end{bmatrix}$. Thus, the vectors in $N(A)$ are of the form $c_1\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0.75 \\ 0 \\ 0.25 \\ -0.25 \\ 1 \end{bmatrix}$ for arbitrary scalars $c_1$ and $c_2$.

(e) It is the number of pivots, 3.

(f) We already have the special solutions, and so we only need to find a particular solution, which we are already given, namely $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. It is important to note that any particular solution works.

We could also get a particular solution by setting the free variables $x_2$ and $x_5$ to 0. In this case we can find the particular solution by inspection, since we know that $A\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = b$ and $A\begin{bmatrix} 0.75 \\ 0 \\ 0.25 \\ -0.25 \\ 1 \end{bmatrix} = 0$, thus $A(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.75 \\ 0 \\ 0.25 \\ -0.25 \\ 1 \end{bmatrix}) = b$ so a particular solution to the system is $\begin{bmatrix} 0.25 \\ 0 \\ 0.75 \\ 0.25 \\ 0 \end{bmatrix}$. Therefore the complete solution is of the form $\begin{bmatrix} 0.25 \\ 0 \\ 0.75 \\ 0.25 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0.75 \\ 0 \\ 0.25 \\ -0.25 \\ 1 \end{bmatrix}$.

**Problem 8** *Monday 2/26*

Suppose the $m \times n$ matrix $R$ is in row reduced echelon form $\begin{bmatrix} I & F \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, with $r$ nonzero rows and first $r$ columns as pivot columns.
(a) Describe the column space and the nullspace of $R$.
(b) Do the same for the $m \times 2n$ matrix $B$=$[R \quad R]$.
(c) Do the same for the $2m \times n$ matrix $C$=$\begin{bmatrix} R \\ R \end{bmatrix}$.
(d) Do the same for the $2m \times 2n$ matrix $D$=$\begin{bmatrix} R & R \\ R & R \end{bmatrix}$.

***Solution 8***

(a) The column space of $R$ consists of all $m$-dimensional vectors such that the last $m-r$ coordinates are 0.
To get the nullspace, notice that there are exactly $r$ pivot variables corresponding to the first $r$ columns, and there are $n-r$ free variables. We must find $n-r$ special solutions by setting all but one of the free variables to 0 and one to 1. then our nullspace is going to be the span of these special solutions.
(b)Clearly, $C(B) = C(R)$.
The only difference from $A$ for the nullspace of $B$ is that now we have $r$ pivot variables corresponding to the first $r$ columns, and there are $2n-r$ free variables.
(c) The column space of $C$ consists of all $2m$-dimensional vectors such that $r+1$, $r+2$, $\ldots$, $m$th and the last $m-r$ coordinates are 0.
The nullspace of $C$ is the same as the nullspace of $R$ since we only have the equations repeated here, which does not affect the solutions to the system.
(d) The column space of $D$ consists of all $2m$-dimensional vectors such that the $r+1$, $r+2$, $\ldots$, $m$th and the last $m-r$ coordinates are 0.
The nullspace of $C$ is the same as the nullspace of $A$ since we only have the equations repeated here, which does not affect the solutions to the system.

**Problem 9** *Monday 2/26*

Do problem 17 of section 3.3 in your book.

***Solution 9***

(a) Call the $i^{th}$ column of $B$, $B_i$. Then the problem states that $B_j = c_1B_1 + \ldots + c_{j-1}B_{j-1}$. Let the columns of $AB$ be denoted by $(AB)_i$. It is clear that $(AB)_j = A \cdot B_j = A \cdot (c_1B_1 + \ldots + c_{j-1}B_{j-1}) = c_1A \cdot B_1 + \ldots + c_{j-1}A \cdot B_{j-1} = c_1(AB)_1 + \ldots + c_{j-1}(AB)_{j-1}$ which proves the claim.
(b) Take $A_1$ to be the identity matrix and $A_2$ to be the all zeros matrix.

**Problem 10** *Monday 2/26*

Suppose the $m \times n$ matrix $A$ ($m < n$) has a *right inverse* $B$, that is, a matrix $B$ such that $AB = I$, the identity.

(a) What must the dimensions (the height and width) of $B$ and of $I$ be?
(b) Try calculating $B$ in `MATLAB`: let $A = \begin{bmatrix} 4 & -3 & 1 \\ -2 & 0 & 2 \end{bmatrix}$ and use `A\I`. (This is the code in `MATLAB` for finding a matrix $B$ such that $AB = I$. The $k \times k$ identity matrix is `eye(k)` in `MATLAB`.)
(c) Now try calculating $B$ another way, with `rref([A I])`. (This is the reduced row echelon form, the result of Gauss-Jordan elimination.) What do you get? Use your result to state another, different, $B$ with $AB = I$. Why is $B$ not unique?
(d) Why can't there be a left inverse $CA = I$?

***Solution 10***

(a) $B$ is n-by-m, and $I$ is m-by-m.
(b) I get $\begin{bmatrix} 0.2 & -0.1 \\ 0 & 0 \\ 0.2 & 0.4 \end{bmatrix}$ for $B$.
(c) I get $\begin{bmatrix} 1 & 0 & -1 & 0 & -0.5 \\ 0 & 1 & -1.6667 & -0.3333 & -0.6667 \end{bmatrix}$ for `rref([A I])`. The last two columns suggest to me that $B = \begin{bmatrix} 0 & -0.5 \\ -0.3333 & -0.6667 \\ 0 & 0 \end{bmatrix}$, and indeed $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
So $B$ is not unique. Let $b_1$ and $b_2$ be the columns of $B$. Then looking for $B$ is the same as solving $Ab_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $Ab_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since $m < n$ we know then that $r < n$ so there are free variables. This means that if there is a solution, this is not unique (because $N(A)$ is not just 0).
(d) $A$ has rank $m$ (or less), $C$ has rank $m$ (or less), so $CA$ has rank at most $m$ also. So it can't equal $I$, which has full rank $n$.
Moral: only square matrices can have both left and right inverses. (And if both inverses exist, they're the same!)