18.06 Problem Set 2 - Solutions

Due Wednesday, Feb. 21, 2007 at **4:00 p.m.** in 2-106

Each problem is worth 10 points. The date next to the problem number indicates the lecture in which the material is covered.

Problem 1 *Monday 2/12*

Use Gauss-Jordan elimination to find the inverse of $A=\begin{bmatrix} 1 & 4 & 2 \\ 1 & 4 & 1 \\ 2 & 10 & 5 \end{bmatrix}$.

Solution 1

$$\begin{bmatrix} 1 & 4 & 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 & 1 & 0 \\ 2 & 10 & 5 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 2 & 10 & 5 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & -2 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -3 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 0 & -1 & 2 & 0 \\ 0 & 2 & 0 & -3 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 5 & 0 & -2 \\ 0 & 2 & 0 & -3 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 5 & 0 & -2 \\ 0 & 1 & 0 & -3/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}$$

Therefore, A^{-1} is $\begin{bmatrix} 5 & 0 & -2 \\ -3/2 & 1/2 & 1/2 \\ 1 & -1 & 0 \end{bmatrix}$

Problem 2 *Monday 2/12*

Do problem 29 of section 2.5 in your book.

Solution 2

(a) True. This matrix has three pivots maximum.
(b) False. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible.
(c) True. A^{-1} has inverse A.
(d) True. A^2 has inverse $A^{-1} * A^{-1}$.

Problem 3 *Monday 2/12*

Do problem 32 of section 2.5 in your book.

Solution 3

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$\rightarrow$ add the second row to the first row $\rightarrow$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$\rightarrow$ add the third row to the second row $\rightarrow$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$\rightarrow$ add the fourth row to the third row $\rightarrow$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Therefore, $A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

For 5 by 5 alternating matrix, the inverse is $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Problem 4 *Wednesday 2/14*

Do problem 11 of section 2.6 in your book.

Solution 4

A is already an upper triangular matrix, so for $A = LU$ decomposition, $L = I$, $U = A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}$.

For $A = LDU$ decomposition, $L = I$,$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$,$U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

Problem 5 *Wednesday 2/14*

Compute the LU decomposition for the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

Solution 5

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 3 & 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 & 1 \end{array}\right)$$

$\rightarrow$ substract the first row from the second, third, and fourth row $\rightarrow$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right)$$

$\rightarrow$ substract the second row from the third and fourth row $\rightarrow$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\rightarrow \text{substract the third row from fourth row} \rightarrow$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right)$$

Therefore, $NA = U$ for

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$A = N^{-1}U$, so

$$L = N^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

gives the LU decomposition of A.

Problem 6 *Wednesday 2/14*

Do problem 16 of section 2.6 in your book.

Solution 6

Let $c = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$.

$Lc = b \leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

Therefore, $c = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

Let $x = \begin{bmatrix} y \\ z \\ w \end{bmatrix}$.

$Lc = b \leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Therefore, $x = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Here, $Ax = LUx = Lc = b$ and $A = LU = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Problem 7 *Friday 2/16*

Let $P = P_{45}P_{16}P_{56}P_{12}P_{34}$, where the P_{ij} are permutation matrices of order 6.

Solution 7

(a) What is $P\begin{bmatrix}1\\2\\3\\4\\5\\6\end{bmatrix}$?

$$P\begin{bmatrix}1\\2\\3\\4\\5\\6\end{bmatrix} = P_{45}P_{16}P_{56}P_{12}P_{34}\begin{bmatrix}1\\2\\3\\4\\5\\6\end{bmatrix} = P_{45}P_{16}P_{56}P_{12}\begin{bmatrix}1\\2\\4\\3\\5\\6\end{bmatrix} = P_{45}P_{16}P_{56}\begin{bmatrix}2\\1\\4\\3\\5\\6\end{bmatrix}$$
$$= P_{45}P_{16}\begin{bmatrix}2\\1\\4\\3\\6\\5\end{bmatrix} = P_{45}\begin{bmatrix}5\\1\\4\\3\\6\\2\end{bmatrix} = \begin{bmatrix}5\\1\\4\\6\\3\\2\end{bmatrix}$$

(b) What is P?
P is the permutation matrix sending
row 1 $\to$ row 2
row 2 $\to$ row 6
row 3 $\to$ row 5
row 4 $\to$ row 3
row 5 $\to$ row 1
row 6 $\to$ row 4
when multiplied on the left.

Therefore $P = \begin{bmatrix}0&0&0&0&1&0\\1&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&1\\0&0&1&0&0&0\\0&1&0&0&0&0\end{bmatrix}$

Problem 8 *Friday 2/16*

Do problem 16 of section 2.7 in your book.

Solution 8

(a) $[A^2 + B^2$ is symmetric$]$
$\leftrightarrow [(A^2 + B^2)^T = A^2 + B^2]$
$\leftrightarrow [(A^2)^T + (B^2)^T = A^2 + B^2\ ]$
$\leftrightarrow [(A^T)^2 + (B^T)^2 = A^2 + B^2]$.
This is true since $A = A^T$ and $B = B^T$.
(b) $[(A + B)(A - B)$ is symmetric$]$
$\leftrightarrow [\{(A + B)(A - B)\}^T = (A + B)(A - B)]$
$\leftrightarrow [(A - B)^T(A + B)^T = (A + B)(A - B)\ ]$
$\leftrightarrow [(A^T - B^T)(A^T + B^T) = (A + B)(A - B)\ ]$
$\leftrightarrow [(A - B)(A + B) = (A + B)(A - B)\ ]$
$\leftrightarrow [A^2 - BA + AB - B^2 = A^2 + BA - AB - B^2]$
$\leftrightarrow [AB = BA\ ]$

This is not true since A and B do not generally commute.
(c) [ABA is symmetric]
$\leftrightarrow$ [$(ABA)^T = ABA$]
$\leftrightarrow$ [$A^T B^T A^T = ABA$]
This is true since $A = A^T$ and $B = B^T$.
(c) [$ABAB$ is symmetric]
$\leftrightarrow$ [$(ABAB)^T = ABAB$]
$\leftrightarrow$ [$B^T A^T B^T A^T = ABAB$]
$\leftrightarrow$ [$BABA = ABAB$]
This is not true since A and B do not generally commute.

Problem 9 *Friday 2/16*

Let $A=\begin{bmatrix} 2 & 6 & 5 \\ 3 & 9 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.
Compute an LU factorization of A if one exists, or find a permutation matrix P such that $PA = LU$ otherwise.

Solution 9

For $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $PA = \begin{bmatrix} 2 & 6 & 5 \\ 1 & 2 & 3 \\ 3 & 9 & 1 \end{bmatrix}$
$\begin{bmatrix} 2 & 6 & 5 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 9 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 6 & 5 & 1 & 0 & 0 \\ 0 & -1 & 1/2 & -1/2 & 1 & 0 \\ 0 & 0 & -13/2 & -3/2 & 0 & 1 \end{bmatrix}$
Therefore, for $N = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix}$, $NPA = \begin{bmatrix} 2 & 6 & 5 \\ 0 & -1 & 1/2 \\ 0 & 0 & -13/2 \end{bmatrix} = U$.
Therefore, $PA = N^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 5 \\ 0 & -1 & 1/2 \\ 0 & 0 & -13/2 \end{bmatrix}$ is the LU decomposition of PA.

Problem 10 *Friday 2/16*

Are the following statements true or false? For each statement explain why it is true or give a counterexample if it is false. All matrices are assumed to be square.

Solution 10

(a) The product of two upper triangular matrices is an upper triangular matrix.
True.
Suppose that A and B is $n \times n$ upper triangular. Then we have $A_{ij} = 0, B_{ij} = 0$ for $i > j$.
$AB_{ij} = \Sigma_{k=1}^{n} A_{ik} B_{kj} = \Sigma_{k=1}^{i} A_{ik} B_{kj} + \Sigma_{k=i+1}^{n} A_{ik} B_{kj}$
For $i > j$, the first term is 0 since $A_{ik} = 0$ for $i > k$, and the second term is 0 since $B_{kj} = 0$ for $k > i > j$. Therefor $AB_{ij} = 0$ for $i > j$, which means AB is upper triangular matrix.
(b) The product of two symmetric matrices is a symmetric matrix.
False.
Even if $A^T = A$, and $B^T = B$ $(AB)^T = B^T A^T = BA$ is not guaranteed to be equal to AB.
(c) The product of two permutation matrices is a permutation matrix.
True.
The composition of two permutations is also a permutation.
(d) The inverse of a lower triangular matrix is lower triangular, if it exists .
True.

If a lower triangilar matrix L has an inverse, then diagonal entries are all nonzero. Therefore, in the Gauss-Jordan elimination, we can take the k-th row as a k-th pivot to get rid of entries in k-th column. Since 1,2,3... (k-1)th row has entry 0 in k-th column, we only get lower trianular matrices as elimination matrices. ¿From the same reasoning in (a), the product of lower triangular matrices is lower trianular. Therefore, we get lower triangular inverse.