18.06 Problem Set 10 - Solutions

Due Friday, May 11, 2007 at **1:00 p.m.** in 2-106

Problem 1 *Wednesday 5/2*

Do problem 7 of section 7.1 in your book.

Solution 1

(a) $T(T(v)) = v$, transformation T^2 is linear.
(b) $T(T(v)) = T(v + (1,1)) = v + (2,2)$, transformation T^2 is not linear since $T(0) \neq 0$.
(c) $T(T((v_1, v_2))) = T((-v_2, v_1)) = (-v_1, -v_2)$, transformation T^2 is linear.
(d) $T(T(v)) = T(\frac{v_1+v_2}{2}, \frac{v_1+v_2}{2}) = (\frac{v_1+v_2}{2}, \frac{v_1+v_2}{2})$, transformation T^2 is linear. Note that projecting twice is the same as projecting once, we get $T^2 = T$.

Problem 2 *Wednesday 5/2*

Do problem 14 of section 7.1 in your book.

Solution 2

(1) Since A is invertible, we can multiply both sides by A^{-1} and we get $M = A^{-1}0 = 0$, so the kernel of T is the zero matrix.
(2) Since A is invertible, given B we can find $M = A^{-1}B$ which will give $AM = B$. This implies that the range of T is the whole space V of all two by two matrices.

Problem 3 *Wednesday 5/2*

Do problem 15 of section 7.1 in your book.

Solution 3

To show that I is not in the range of T we need to show that there is no matrix M such that $AM = I$. If there existed such an M it would be the inverse of A. But A is not invertible so no such M exists.
The matrix $M = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$ is a nonzero matrix, and $T(M) = AM = 0$.

Problem 4 *Wednesday 5/2*

Do problem 18 of section 7.1 in your book.

Solution 4

After calculation one obtains that $T(M) = \begin{bmatrix} 0 & m_{12} \\ 0 & 0 \end{bmatrix}$, so, for example, the matrix $M = \begin{bmatrix} 3 & 9 \\ 2 & 1 \end{bmatrix}$ yields $T(M) \neq 0$.
From $T(M) = \begin{bmatrix} 0 & m_{12} \\ 0 & 0 \end{bmatrix}$ it is clear that the kernel of T consists of matrices of the form $\begin{bmatrix} m_{11} & 0 \\ m_{21} & m_{22} \end{bmatrix}$, where m_{11}, m_{21}, m_{22} are arbitrary, and the range of T consists of all matrices $\begin{bmatrix} 0 & m_{12} \\ 0 & 0 \end{bmatrix}$, where m_{12} is arbitrary.

Problem 5 *Wednesday 5/7*

(a) For those transformations in problem 7 of section 7.1 which are linear, find the matrix that represents them when we take the basis $\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$ for the input and the output spaces.
(b) For these transformations, find (if possible) a basis so that the matrix that represents the transformation is diagonal. (*Note:* we want the same basis for the input and the output).

Solution 5

(a) For (a) the matrix is $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$, for (c) the matrix is $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$, for (d) the matrix is $\begin{bmatrix}1/2 & 1/2\\1/2 & 1/2\end{bmatrix}$.
(b) For part (a) the matrix is already diagonal.
Part (c) there is no basis of $\mathbb{R}^2$ that makes the matrix diagonal, as in order to do that we would need to be able to diagonalize the matrix with eigenvectors in $\mathbb{R}^2$, and the matrix has complex eigenvectors. (Recall that when we change the basis we get a similar matrix).
For (d) we take the basis $w_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $w_2 = \begin{bmatrix}1\\-1\end{bmatrix}$ (not that these are the eigenvectors of the projection). Then $T(w_1) = w_1$ and $T(w_2) = 0$ so the matrix of the transformation is $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$.
Note that this is exactly what we get when we diagonalize the matrix $\begin{bmatrix}1/2 & 1/2\\1/2 & 1/2\end{bmatrix}$.

Problem 6 *Wednesday 5/7*

Do problem 14 of section 7.2 in your book.

Solution 6

(a) $T(v_1, v_2) = -(v_2, v_1)$ works.
(b) $T(v_1, v_2) = (\frac{v_1+v_2}{2}, \frac{v_1+v_2}{2})$ (a projection) works.
(c) Suppose there was a transformation T such that $T = T^{-1}$ and $T = T^2$. Then $T = T^2 = T \circ T = T \circ T^{-1}$ and the latter is the identity. Therefore only the identity transformation can satisfy both being its own inverse and equal its square.

Problem 7 *Wednesday 5/7*

Consider a linear transformation $T : \mathbb{R}^3 \to: \mathbb{R}^3$ such that $T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = \begin{bmatrix}3\\6\\9\end{bmatrix}$, $T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}4\\2\\6\end{bmatrix}$ and $T\left(\begin{bmatrix}2\\2\\5\end{bmatrix}\right) = \begin{bmatrix}6\\6\\15\end{bmatrix}$.
(a) Write down the matrix A_T corresponding to T in the basis $v_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $v_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}$ and $v_3 = \begin{bmatrix}2\\2\\5\end{bmatrix}$ for both input and output spaces.
(b) Write the matrix M that changes the basis of $\mathbb{R}^3$ from the v-basis to the standard basis.
(c) Write down the matrix B_T corresponding to T in the standard basis for both input and output spaces.
(d) How are A_T and B_T related? What are their eigenvalues?

Solution 7

(a) The information in the basis vectors v_1, v_2, v_3 says that $T(v_1) = 3v_1$, $T(v_2) = v_1 + 3v_2$ and $T(v_3) = 3v_3$. So we have $A_T \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}3\\0\\0\end{bmatrix}$, $A_T \cdot \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\3\\0\end{bmatrix}$, $A_T \cdot \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\3\end{bmatrix}$, from which we get that $A_T = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(b) The matrix is $M = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \\ 3 & 1 & 5 \end{bmatrix}$.

(c) Since what is happening here is changing the basis in which we are representing the linear transformation, we have that $B_T = MA_TM^{-1} = \begin{bmatrix} 5 & 1/2 & -1 \\ 4 & 4 & -2 \\ 6 & 3/2 & 0 \end{bmatrix}$.

(d) The matrices A_T and B_T are similar, $A_T = M^{-1}B_TM$. Their eigenvalues are $3, 3, 3$.

Problem 8 *Wednesday 5/7*

Do problem 32 of section 7.2 in your book.

Solution 8

Multiplying the given matrices results in the matrix
$ST = \begin{bmatrix} \cos 2\alpha \cos 2\theta + \sin 2\alpha \sin 2\theta & \sin 2\alpha \cos 2\theta - \cos 2\alpha \sin 2\theta \\ \cos 2\alpha \sin 2\theta - \sin 2\alpha \cos 2\theta & \sin 2\alpha \sin 2\theta + \cos 2\alpha \cos 2\theta \end{bmatrix} = \begin{bmatrix} \cos 2(\theta - \alpha) & -\sin 2(\theta - \alpha) \\ \sin 2(\theta - \alpha) & \cos 2(\theta - \alpha) \end{bmatrix}$, thus the composition of these two reflections gives a rotation by $2(\theta - \alpha)$.