# 18.06 Problem Set 1 - Solutions

Due Wednesday, Feb. 14, 2007 at **4:00 p.m.** in 2-106

Each problem is worth 10 points. The date next to the problem number indicates the lecture in which the material is covered.

**Problem 1** *Wednesday 2/07*

Consider the following system of equations:

$$\begin{aligned} x + 3y + 2z &= 6 \\ 2x + 5y + 4z &= 1 \\ 3x + 8y + 6z &= 7 \end{aligned}$$

What do you notice about the equations?
The first two planes intersect in a line. What do you know about that line and the third plane? How many solutions does the system have?

***Solution 1***

If you add the first and second equations, you obtain the third equation.
This implies that if $(x, y, z)$ satisfies the first two equations (i.e., if it is a point in the line where the first two planes intersect) then it also satisfies the third equation (i.e., it is contained in the third plane). Thus, that line is contained in the third plane.
This system has infinitely many solutions, all the points in the common line.

**Problem 2** *Wednesday 2/07*

(a) Find a matrix $A$ such that $A\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

(b) What is $A\begin{bmatrix} 3 \\ 3 \end{bmatrix}$?

***Solution 2***

(a) Let us name the first and second columns of $A$, $a_1$ and $a_2$, respectively. Then, the first equation means that we have the linear combination $2 \times a_1 + 0 \times a_2 = (6, 10)$. So, the first column must be $(3, 5)$. Now, the second equation means that we have the linear combination $1 \times a_1 + 3 \times a_2 = (-3, 2)$, which yields $a_2 = (-2, -1)$.
$A = \begin{bmatrix} 3 & -2 \\ 5 & -1 \end{bmatrix}$.
(b) $A\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 5 \end{bmatrix} + 3\begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$.

**Problem 3** *Wednesday 2/07*

Do problem 26 of section 2.1 in your book.

***Solution 3***

`A*v = ones(4,4)* ones(4,1)`$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$

B = eye(4)+ones(4,4)$=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}+\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}=\begin{bmatrix}2&1&1&1\\1&2&1&1\\1&1&2&1\\1&1&1&2\end{bmatrix}$

v = zeros(4,1)+2* ones(4,1)$=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+2\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}2\\2\\2\\2\end{bmatrix}$

B*v$=\begin{bmatrix}10\\10\\10\\10\end{bmatrix}$

**Problem 4** *Wednesday 2/07*

Let's practice using Matlab to check that in general $AB$ and $BA$ are not equal. (Hint: you can type `diary` at the beginning of your session to save a transcript.)
Let's start with matices of different sizes. Let `A=ones(3,2)` and `B=ones(2,3)` (that is, the 3-by-2 and 2-by-3 matrices with all entries equal to 1). Compute $AB$ and $BA$. What are their sizes?
Now, let's multiply to 3-by-3 matrices. Let `C=[a b c; d e f; g h i]`, where `a...i` are nine of your favorite numbers. *Now let the computer pick one:* `D=rand(3,3)` gives us a random 3-by-3 matrix. What are $CD$ and $DC$? Are they equal?

***Solution 4***

```
>>A=ones(3,2)
A=
    1    1
    1    1
    1    1

>>B=ones(2,3)
B=
    1    1    1
    1    1    1

>>A*B
ans=
    2    2    2
    2    2    2
    2    2    2

>>B*A
ans=
    3    3
    3    3
```

So, `A*B` is a 3-by-3 matrix, whereas `B*A` is a 2-by-2 matrix.

```
>>C=[1,2,3;4,5,6;7,8,9]
C=
    1    2    3
    4    5    6
    7    8    9
```

```
>>D=rand(3,3)
D=
    0.9501   0.4860   0.4565
    0.2311   0.8913   0.0185
    0.6068   0.7621   0.8214

>>C*D
ans=
    3.2329   4.5549   2.9577
    8.5973  10.9730   6.8468
   13.9616  17.3911  10.7360

>>D*C
ans=
    6.0893   7.9819   9.8745
    3.9259   5.0668   6.2077
    9.4051  11.5954  13.7858
```

Now, `C*D` and `D*C` are both 3-by-3 matrixes, but they are not equal. We can see that in general, matrix multiplication is not commutative.

**Problem 5** *Friday 2/09*

Write examples of systems $A\vec{x} = \vec{b}$ where $A$ is a 3-by-3 matrix and:

1. the three planes meet in a common line
2. in the row picture, all three planes are parallel but distinct
3. the intersection of the first two planes does not intersect the third plane
4. $\vec{b}$ is not a linear combination of the columns of $A$.
5. in the column picture, $\vec{b}$ is a multiple of the second column of $A$.

***Solution 5***

Answers may vary, but here are some examples:

1. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$
3. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$
4. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 1 & -1 & -3 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$
5. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ -1 & -1 & 5 \end{bmatrix} \vec{x} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$

**Problem 6** *Friday 2/09*

Answer the following questions for the systems in problem 5:
(a) How many solutions does each have? Describe the shape (point, line, ...) of each solution set.
(b) Reduce each by elimination (you need not back-substitute) and check your answer.

***Solution 6***

(a) Counting solutions:

1. Here, the solution set is the common line (infinitely many points).
2. There is no solution, since the planes never meet.
3. There is no solution, since there is no point in which the three planes all meet.
4. There is no solution, since no linear combination of columns $A\vec{x}$ ever yields $\vec{b}$.
5. The number of solutions depends on *how many* ways we can form $\vec{b}$ as a linear combination $A\vec{x}$ of columns of $A$. (So there's at least one solution, but there could be more if your matrix $A$ was singular.)

(b) Elimination:

1. $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -1 & -1 & 1 \\ 1 & 0 & 1 & 7 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; the line defined by the first two planes $x + 2y + 3z = 5$ and $-y - z = 1$ is the line $\vec{x} = \vec{a} + t\vec{r}$, where $\vec{a} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{r} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

2. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ which clearly has no solution.
   (the second row says $0 \times x + 0 \times y + 0 \times z = 1$, which is impossible)

3. $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & -3 \\ 1 & 1 & 0 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & -3 \\ 0 & 1 & 0 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ which has no solution.

4. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 \\ 1 & -1 & -3 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ has no solution.

5. For this example, $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 2 \\ -1 & -1 & 5 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 2 \\ 0 & 1 & 8 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 4 & 0 \end{bmatrix}$; this example is nonsingular, with the unique solution $\vec{x} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.

**Problem 7** *Friday 2/09*

Solve the following system by elimination and back substitution:

$$\begin{aligned} 2x + 3y + z &= 0 \\ x - 2y - z &= -3 \\ x + y + 2z &= 3 \end{aligned}$$

Write down the elimination matrices $E_{21}$, $E_{31}$, $E_{32}$ you used.

***Solution 7***

We start by doing elimination:

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & -2 & -1 & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & -\frac{7}{2} & -\frac{3}{2} & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & -\frac{7}{2} & -\frac{3}{2} & -3 \\ 0 & -\frac{1}{2} & \frac{3}{2} & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & -\frac{7}{2} & -\frac{3}{2} & -3 \\ 0 & 0 & \frac{12}{7} & \frac{24}{7} \end{bmatrix}$$

The elimination matrices used were:

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix};$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{7} & 1 \end{bmatrix}.$$

Then we proceed to back substitution:

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & -\frac{7}{2} & -\frac{3}{2} & -3 \\ 0 & 0 & \frac{12}{7} & \frac{24}{7} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & -\frac{7}{2} & -\frac{3}{2} & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & -\frac{7}{2} & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 3 & 0 & -2 \\ 0 & -\frac{7}{2} & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 2 & 3 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution of the system is, then, $x = -1$, $y = 0$, $z = 2$.

**Problem 8** *Monday 2/12*

Consider the matrices $A = \begin{bmatrix} 5 & -3 & -9 \\ 2 & 4 & -1 \\ -1 & 7 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \\ -1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ -3 & 3 \end{bmatrix}$.

(a) Find $AB$ and $AC$.

(b) Do you notice anything special? Why does this tell you $A$ is not invertible?

***Solution 8***

(a) $AB = \begin{bmatrix} 29 & -20 \\ 9 & 5 \\ -9 & 20 \end{bmatrix}$ ... and so is $AC$.

(b) They're equal! $A$ is not invertible because if it had an inverse, then $A^{-1}AB = A^{-1}AC$ would mean $B = C$. Alternatively, you can think that there are two distinct vectors $\vec{x}$and $\vec{y}$such that $A\vec{x} = A\vec{y}$. Just take $\vec{x}$to be the first column of $B$and $\vec{y}$to be the first column of $C$. Then, both $A\vec{x}$and $A\vec{y}$are equal to the first column of $AB = AC$, and as you know, this imples that $A$is not invertible.

**Problem 9** *Monday 2/12*

Do problem 13 of section 2.4 in your book.

***Solution 9***

We want to know which matrices are guaranteed to be equal to $(A - B)^2$. It might help to keep in mind that the expansion of $(A - B)^2$ is $(A - B)^2 = A^2 - AB - BA + B^2$.

- In general, $A^2 - B^2 \neq (A - B)^2$. For example, take $A = 2I$ to be twice the identity matrix, and $B = I$ the identity matrix. Then $A^2 - B^2 = 4I^2 - I^2 = 3I$ but $(A - B)^2 = I$.
- $(B - A)^2 = (-I(A - B))^2 = (-I)^2(A - B)^2 = (A - B)^2$
- In general, $A^2 - 2AB + B^2 \neq (A - B)^2$. Any two matrices $A$ and $B$ such that $AB \neq BA$ serve as a

counterexample.

- $(A-B)^2 = A(A-B) - B(A-B)$, by distributivity.
- $(A-B)^2 = A^2 - AB - BA + B^2$, again by distributivy from line above (this is how we get the expansion in the first place).

**Problem 10** *Monday 2/12*

Do problem 7 of section 2.5 in your book.

***Solution 10***

We want to show that if $A$ is a 3-by-3 matrix such that row 1+ row 2 = row 3, then $A$ is not invertible. Let us name the three rows of $A$ as $a_1$, $a_2$ and $a_3$.
(a)Suppose that $A \cdot \vec{x} = (1,0,0)$. This means that $a_1 \cdot \vec{x} = 1$, $a_2 \cdot \vec{x} = 0$ and $a_3 \cdot \vec{x} = 0$. But this cannot happen, since

$$a_3 \cdot \vec{x} = (a_1 + a_2) \cdot \vec{x} = a_1 \cdot \vec{x} + a_2 \cdot \vec{x} = 1 + 0 = 0.$$

(b)By the same reasoning as above, if $A \cdot \vec{x} = (b_1, b_2, b_3)$ then we must have $b_3 = b_1 + b_2$:

$$b_3 = a_3 \cdot \vec{x} = (a_1 + a_2) \cdot \vec{x} = a_1 \cdot \vec{x} + a_2 \cdot \vec{x} = b_1 + b_2.$$

So, not all vectors $\vec{b}$will have a solution for the system $A\vec{x}=\vec{b}$, which, as we know, implies that $A$is not invertible (for invertible matrices there is always a solution, by doing $\vec{x}=A^{-1}\vec{b}$).
(c) When doing elimination on the matrix $A$, the following happens:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 - a_1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 - a_1 - a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}$$

Since elimination yields a row of zeros, the matrix $A$ is not invertible.