18.06 **QUIZ 3** May 07, 2007

Your PRINTED name is: SOLUTIONS

Please circle your recitation:

(1)	M 2	2-131	A. Osorno
(2)	M 3	2-131	A. Osorno
(3)	M 3	2-132	A. Pissarra Pires
(4)	T 11	2-132	K. Meszaros
(5)	T 12	2-132	K. Meszaros
(6)	T 1	2-132	Jerin Gu
(7)	T 2	2-132	Jerin Gu

Grading
1
2
3
4
Total:

Problem 1 *(25 points)*

(a) Compute the singular value decomposition $A = U\Sigma V^T$ for $A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}$.

(b) Find orthonormal bases for all four fundamental subspaces of A.

Solution 1

(a) $A^TA = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, with eigenvalues 3 and 1. Thus $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$.

v_1 is a normal eigenvector corresponding to 3, so $v_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$.

v_2 is a normal eigenvector corresponding to 1, so $v_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

$u_1 = \frac{1}{\sqrt{3}}Av_1 = \begin{pmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$; $u_2 = Av_2 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$.

For u_3 we find a basis for the nullspace of A^T: $u_3 = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$.

Thus $A = \underbrace{\begin{pmatrix} -1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{V^T}$.

(b) Rowspace: v_1 and v_2, i.e. $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$. Nullspace: 0.

Columnspace: u_1 and u_2, i.e. $\begin{pmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$. Left nullspace: u_3, i.e. $\begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$.

Problem 2 *(25 points)*

(a) Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$.

(b) Find 3 linearly independent eigenvectors of A.

(c) Write down a diagonal matrix that is similar to A.

(d) Diagonalize the matrix A as $A = Q\Lambda Q^T$ with orthogonal matrix Q.

Solution 2

(a) Notice that $A^T = A$ and $A^2 = A^T A = 9I$. So if λ is an eigenvalue of A, $\lambda^2 = 9$. Thus $\lambda = \pm 3$.

The trace of A is 3, so $\lambda_1 + \lambda_2 + \lambda_3 = 3$. We get then that $\lambda_1 = 3$, $\lambda_2 = 3$ and $\lambda_3 = -3$.

(b) For $\lambda = 3$, we need to find vectors in the nullspace of $\begin{pmatrix} -2 & 2 & 2 \\ 2 & -2 & -2 \\ 2 & -2 & -2 \end{pmatrix}$: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

For $\lambda = -3$, we find the nullspace of $\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & -2 \\ 2 & -2 & 4 \end{pmatrix}$: $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

(c) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ since A is diagonalizable.

(d) We need to find orthonormal eigenvectors. We can do this by Gram-Schmidt or by inspection.

$$A = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}}_{Q^T}.$$

Problem 3 *(25 points)*

Consider the matrix $A = \begin{pmatrix} a & 2 & 1 \\ 2 & a & 1 \\ 1 & 1 & 2 \end{pmatrix}$, where a is a real number.

(a) For which values of the parameter a is the matrix A positive definite?

(b) For which values of the parameter a is the matrix $-A$ positive definite?

(c) For which values of the parameter a is the matrix A singular?

Solution 3

(a) We will use the upper left determinants:

$a > 0$

$a^2 - 4 > 0 \Rightarrow a > 2$

$2(a+1)(a-2) > 0$ which is always true if $a > 2$.

So the only condition we have is $a > 2$.

(b) Again using upper left determinants:

$-a > 0 \Rightarrow a < 0$

$a^2 - 4 > 0 \Rightarrow a < -2$

$-2(a+1)(a-2) > 0$ which is never true if $a < -2$.

So $-A$ is never positive definite.

(c) $det(A) = 2(a+1)(a-2)$, so A is singular if $a = -1$ or $a = 2$.

Problem 4 *(25 points)*

(a) Find the steady state for the Markov matrix $A = \begin{pmatrix} 0.2 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{pmatrix}$.

(b) Calculate the limit of $A^n \begin{pmatrix} 0 \\ 20 \\ 0 \end{pmatrix}$ as $n \to \infty$.

Solution 4

(a) The steady state is the eigenvector corresponding to the eigenvalue 1. As a convention, we take it so that the sum of the components is 1.

To find it, we need to look at the nullspace of the matrix $A - I = \begin{pmatrix} -0.8 & 0.4 & 0.3 \\ 0.4 & -0.8 & 0.3 \\ 0.4 & 0.4 & -0.6 \end{pmatrix}$.

The steady state is $\begin{pmatrix} 0.3 \\ 0.3 \\ 0.4 \end{pmatrix}$.

(b) The limit of $A^n \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ as $n \to \infty$ is the steady state, so the limit of $A^n \begin{pmatrix} 0 \\ 20 \\ 0 \end{pmatrix}$ as $n \to \infty$ is $20 \cdot \begin{pmatrix} 0.3 \\ 0.3 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 8 \end{pmatrix}$.