18.06 **QUIZ 1** March 05, 2007

Your PRINTED name is: SOLUTIONS

Please circle your recitation:

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Grading

1
2
3
4
5
Total:

Problem 1 *(20 points)*

Are the following sets of vectors in $\mathbb{R}^3$ vector subspaces? Explain your answer.

(a) vectors $(x, y, z)^T$ such that $2x - 2y + z = 0$ **<u>YES</u>** NO

It is given by a linear equation equal to 0. You can also think about it as the nullspace of the matrix $\begin{pmatrix} 2 & -2 & 1 \end{pmatrix}$.

(b) vectors $(x, y, z)^T$ such that $x^2 - y^2 + z = 0$ YES **<u>NO</u>**

The vector $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is in the set, but if you multiply by -1, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is not.

(c) vectors $(x, y, z)^T$ such that $2x - 2y + z = 1$ YES **<u>NO</u>**

It is given by a linear equation <u>not</u> set equal to 0. In particular, it doesn't contain the 0 vector.

(d) vectors $(x, y, z)^T$ such that $x = y$ AND $x = 2z$ **<u>YES</u>** NO

It is the intersection of two planes! We can think about this set as the nullspace of the matrix $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -2 \end{pmatrix}$.

(e) vectors $(x, y, z)^T$ such that $x = y$ OR $x = 2z$ YES **<u>NO</u>**

It is the union of two planes! Take for example $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ which is not in the set.

Problem 2 *(20 points)*

Let A be a 4×3 matrix with linearly independent columns.

(a) What are the dimensions of the four fundamental subspaces $C(A)$, $N(A)$, $C(A^T)$, $N(A^T)$?

(b) Describe explicitly the nullspace $N(A)$ and the row space $C(A^T)$ of A.

(c) Suppose that B is a 4×3 matrix such that the matrices A and B have exactly the same column spaces $C(A) = C(B)$ and the same nullspaces $N(A) = N(B)$.

Are you sure that in this case $A = B$? YES NO

Prove that $A = B$ or give a counterexample where $A \neq B$.

Solution 2

(a) The columns are linearly independent, so the rank of the matrix is 3. Then $dim C(A) = 3$, $dim C(A^T) = 3$, $dim N(A) = 0$, $dim N(A^T) = 3$.

(b) Since $C(A^T)$ is a 3-dimensional subspace of $\mathbb{R}^3$, it is all of $\mathbb{R}^3$.

(c) The answer is **NO**, for example $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

Problem 3 *(20 points)*

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 5 \\ 1 & 3 & 5 & 9 \end{pmatrix}$$

(a) What is the rank of A?

(b) Find a matrix B such that the column space $C(A)$ of A equals the nullspace $N(B)$ of B.

(c) Which of the following vectors belong(s) to the column space $C(A)$:

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \\ 4 \\ 8 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad ?$$

Solution 3

We will eliminate the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & b_1 \\ 1 & 2 & 3 & 5 & b_2 \\ 1 & 3 & 5 & 9 & b_3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 & b_1 \\ 0 & 1 & 2 & 4 & b_2 - b_1 \\ 0 & 2 & 4 & 8 & b_3 - b_1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 & b_1 \\ 0 & 1 & 2 & 4 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{pmatrix}$$

(a) The rank is 2.

(b) We see from the last row of the reduced matrix that the condition for a vector to be in the column space is $b_1 - 2b - 2 + b_3 = 0$. Thus $C(A)$ is $N(B)$ for

$$B = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}.$$

(c) The last two vectors can't belong to the column space because they are in $\mathbb{R}^4$. From the condition mentioned in part (b), we see that $\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ is in the column space, but $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is not.

Problem 4 *(20 points)*

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & k \end{pmatrix}$$

(a) For which values of k will the system $A\mathbf{x} = (2,3,7)^T$ have a unique solution?

(b) For which values of k will it have an infinite number of solutions?

(c) For $k = 4$, find the LU-decomposition of A.

(d) For all values of k, find the complete solution to the system $A\mathbf{x} = (2,3,7)^T$.
(You might need to consider several cases.)

Solution 4

We will eliminate the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 3 \\ 3 & 4 & k & 7 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & k-3 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k-5 & 0 \end{pmatrix}$$

(a) and (b) We see from this that no matter what k is there is always at least one solution (there is only a potentially 0 row in the eliminated matrix, and we get a 0 in the augmented vector). We could have seen that by inspection from the original matrix, since $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$.

For $k \neq 5$, the matrix has rank 3, so there is a unique solution. For $k = 5$ the matrix has rank 2, so there are infinitely many solutions.

(c) $L=\begin{pmatrix}1&0&0\\1&1&0\\3&1&1\end{pmatrix}$ using the multipliers, and $U=\begin{pmatrix}1&1&1\\0&1&2\\0&0&-1\end{pmatrix}$ from the elimination above.

(d) As noted above, for $k \neq 5$ there is a unique solution, given by $\begin{pmatrix}1\\1\\0\end{pmatrix}$. We can get this from the eliminated matrix, or as mentioned above, by inspection.

For $k = 5$, $\begin{pmatrix}1\\1\\0\end{pmatrix}$ is a particular solution; the general soltuion is given by adding vectors in the nullspace:

$$\begin{pmatrix}1\\1\\0\end{pmatrix} + c\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

.

Problem 5 *(20 points)*

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & -1 & 0 & 0 \\ 2 & 4 & 0 & 4 & 4 \end{pmatrix}$$

(a) Find a basis of the column space $C(A)$.

(b) Find a basis of the nullspace $N(A)$.

(c) Find linear conditions on b_1, b_2, b_3, b_4 that guarantee that the system $A\mathbf{x} = (b_1, b_2, b_3, b_4)^T$ has a solution.

(d) Find the complete solution for the system $A\mathbf{x} = (0, 1, 0, 2)^T$.

Solution 5

You could find the following answers by eliminating, but also, by inspection.

(a) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$

(c) $b_3 - b_1 = 0$ and $b_4 - 2b_2 = 0$.

(d) The general solution is:

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$