18.06 Spring 2006 - Exam 3 Review Problems

SOLUTIONS TO SELECTED PROBLEMS

1. Section 6.1, Problem 4

Answer: A has $\lambda_1 = -3$ and $\lambda_2 = 2$ (check trace and determinant) with $\mathbf{x_1} = (3, -2)$ and $\mathbf{x}_2 = (1, 1)$. A^2 has the same eigenvectors as A, with eigenvalues $\lambda_1 = 9$ and $\lambda_2 = 4$.

2. Section 6.1, Problem 5

Answer: A and B both have $\lambda_1 = 1$ and $\lambda_2 = 1$. $A+B$ has $\lambda_1 = 1$ and $\lambda_2 = 3$. Eigenvalues of $A + B$ **are not** equal to eigenvalues of A plus eigenvalues of B.

3. Section 6.1, Problem 25

Answer: $\lambda = 0, 0, 6$ with eigenvectors $\mathbf{x}_1 = (0, -2, 1)$, $\mathbf{x}_2 = (1, -2, 0)$ and $\mathbf{x}_3 = (1, 2, 1)$.

4. Section 6.2, Problem 2

Answer: If $A = S\Lambda S^{-1}$, then $A^3 = S\Lambda^3 S^{-1}$ and $A^{-1} = S\Lambda^{-1}S^{-1}$.

5. Section 6.2, Problem 18

Answer: The rank of $A - 3I$ is one, hence A is not diagonalizable (3 is a repeated eigenvalue but has only one associated eigenvector). Change any entry except $a_{12} = 1$ to make A diagonalizable.

6. Section 6.3, Problem 6

Answer: $\lambda_1 = 0$ and $\lambda_2 = 2$. Now $v(t) = 20 + 10e^{2t} \to \infty$ as $t \to \infty$.

7. Section 6.4, Problem 7

Answer:

a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has $\lambda = -1, 3$.

b) A has a negative pivot because the pivots have the same signs as the λ's.

c) A can't have two negative eigenvalues because its trace is positive.

8. Section 6.4, Problem 18

Answer: Suppose $A = A^T$ and $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = 0\mathbf{y}$. Then $\mathbf{y}$ is in the nullspace and $\mathbf{x}$ is in the column space. Since $A = A^T$, the column space equals the row space, hence $\mathbf{x}$ is in the row space of A. The row space and nullspace are orthogonal subspaces, so $\mathbf{y} \perp \mathbf{x}$.

If the second eigenvalue is a nonzero number β, then shift by β: $(A - \beta I)\mathbf{x} = (\lambda - \beta)\mathbf{x}$ and $(A - \beta I)\mathbf{y} = \mathbf{0}$ and again $\mathbf{x} \perp \mathbf{y}$.

9. Section 6.4, Problem 22

Answer: If A is skew-symmetric, then $A^T = -A$ and $A^T A = AA^T = -A^2$. Every orthogonal matrix is normal because $A^T A = AA^T = I$.

$A = \begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix}$ is normal if $a = d$.

10. Section 6.5, Problem 1

Answer: A_4 has two positive eigenvalues because $a = 1$ and $det(A_4) = 1$. $\mathbf{x}^T A_1 \mathbf{x}$ is zero for $\mathbf{x} = (1, -1)$ and $\mathbf{x}^T A_1 \mathbf{x} < 0$ for $\mathbf{x} = (6, -5)$.

11. Section 6.5, Problem 16

Answer: $\mathbf{x}^T A\mathbf{x}$ is not positive when $\mathbf{x} = (0,1,0)$ because of the zero on the diagonal.

12. Section 6.6, Problem 2

Answer: If $C = F^{-1}AF$ and also $C = G^{-1}BG$ then $M = FG^{-1}$ gives $B = M^{-1}AM$. If C is similar to A and also to B then A is similar to B.

13. Section 6.7, Problem 1

Answer: $A^T A = \begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix}$ has $\sigma_1^2 = 85$, $\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 4/\sqrt{17} \\ -1/\sqrt{17} \end{bmatrix}$.

14. Section 6.7, Problem 3

Answer: $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ for the column space; $\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{bmatrix}$ for the row space; $\mathbf{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$ for the nullsapce; $\mathbf{v}_2 = \begin{bmatrix} 4/\sqrt{17} \\ -1/\sqrt{17} \end{bmatrix}$ for the left nullspace.

15. Section 6.7, Problem 10

Answer: $A = W\Sigma W^T = U\Sigma V^T$.

16. Section 6.7, Problem 12

Answer: Since $A = A^T$, we have $\sigma_1^2 = \lambda_1^2$ and $\sigma_2^2 = \lambda_2^2$. So $\sigma_1 = 3$ and $\sigma_2 = 2$ (singular values are positive). The unit eigenvectors of $A^T A = AA^T$ are the same as those for A: $\mathbf{u}_1 = \mathbf{v}_1$ and $\mathbf{u}_2 = -\mathbf{v}_2$ (notice the sign change because $\sigma_2 = -\lambda_2$).

17. Section 6.7, Problem 15

Answer:

a) If A changes to $4A$, multiply Σ by 4.

b) $A^T = V\Sigma^T U^T$. And if A^{-1} exists, it is square and equal to $(V^T)^{-1}\Sigma^{-1}U^{-1}$.

18. Section 8.3, Problem 1

Answer: $\lambda = 1$ and $.75$; steady state eigenvector $\mathbf{x} = (.6, .4)$.

19. Section 8.3, Problem 9

Answer: $\mathbf{u}_1 = P\mathbf{u}_0 = (0, 0, 1, 0)$; $\mathbf{u}_2 = P\mathbf{u}_1 = (0, 1, 0, 0)$; $\mathbf{u}_3 = P\mathbf{u}_2 = (1, 0, 0, 0)$; $\mathbf{u}_4 = P\mathbf{u}_3 = \mathbf{u}_0$.

The four eigenvalues of P are $1, i, -1, -i$.