18.06 Spring 2006 - Problem Set 9

SOLUTIONS TO SELECTED PROBLEMS

1. Section 6.6, Problem 4

Answer: A has no repeated eigenvalues so it can be diagonalized: $A = S\Lambda S^{-1}$ makes it similar to Λ.

2. Section 6.6, Problem 5

Answer:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \text{ are similar;}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ by itself and } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ by itself.}$$

3. Section 6.6, Problem 12

Answer: If $M^{-1}JM = K$ then

$$JM = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} = MK = \begin{bmatrix} 0 & m_{12} & m_{13} & 0 \\ 0 & m_{22} & m_{23} & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & m_{42} & m_{43} & 0 \end{bmatrix}.$$

That means $m_{21} = m_{22} = m_{23} = m_{24} = 0$ and M is not invertible.

4. Section 6.6, Problem 20

Answer:

a) If A is similar to B then A^2 is similar to B^2 because $A = M^{-1}BM \Rightarrow$ $A^2 = (M^{-1}BM)(M^{-1}BM) = M^{-1}B^2M$.

b) A^2 and B^2 can be similar when A and B are not similar:

A is the 2×2 zero matrix, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. $A^2 = B^2 =$ zero matrix but A and B are not similar because B has only one eigenvector.

c) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ because

$\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ is diagonalizable to $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ since $\lambda_1 \neq \lambda_2$.

d) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is not similar to $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ because

$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ has only one eigenvector, so not diagonalizable.

e) If we exchange rows 1 and 2 of A, and then exchange columns 1 and 2, the eigenvalues stay the same:

Let P be the permutation matrix that exchanges the first two rows. Then PAP^T is similar to A. (PAP^T is the result of exchanging the rows and then the columns of A.)

5. Section 6.7, Problem 1

Answer:

$$A^TA = \begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix} \text{ has } \sigma_1^2 = 85,\ \mathbf{v_1} = \begin{bmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{bmatrix} \text{ and } \mathbf{v_2} = \begin{bmatrix} 4/\sqrt{17} \\ -1/\sqrt{17} \end{bmatrix}.$$

6. Section 6.7, Problem 7

Answer:

$$AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ has } \sigma_1^2 = 3 \text{ with } \mathbf{u_1} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \text{ and } \sigma_2^2 = 1 \text{ with } \mathbf{u_2} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ has } \sigma_1^2 = 3 \text{ with } \mathbf{v_1} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix};\ \sigma_2^2 = 1 \text{ with } \mathbf{v_2} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \text{ and } \mathbf{v_3} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

Then $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} \end{bmatrix}^T.$

7. Section 6.7, Problem 9

Answer: $A = 12\mathbf{u}\mathbf{v}^T$. Its only singular value is $\sigma_1 = 12$.

8. Section 6.7, Problem 13

Answer: Suppose $A = QR$ and the SVD of R is $R = U\Sigma V^T$. Then multiply by Q. So the SVD of A is $(QU)\Sigma V^T$.