18.06 Spring 2006 - Problem Set 8

SOLUTIONS TO SELECTED PROBLEMS

1. Section 6.3, Problem 3

Answer:

$$\begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}. \text{ Then } \lambda = \tfrac{1}{2}(5 \pm \sqrt{41}).$$

2. Section 6.3, Problem 23

Answer: $A^2 = A$ so $A^3 = A$.

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & 3t \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} t^2 & 3t^2 \\ 0 & 0 \end{bmatrix} + \ldots = \begin{bmatrix} e^t & 3(e^t - 1) \\ 0 & 1 \end{bmatrix}$$

3. Section 6.4, Problem 4

Answer: $\lambda = 10, -5$ with eigenvectors $(1, 2)$ and $(2, -1)$.

$$Q = \tfrac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

4. Section 6.4, Problem 6

Answer: $\lambda = 0, 25$ with eigenvectors $(.8, -.6)$ and $(.6, .8)$.

$$Q = \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix} \text{ or } \begin{bmatrix} -.8 & .6 \\ .6 & .8 \end{bmatrix} \text{ or exchange columns.}$$

5. Section 6.5, Problem 7

Answer:

$A^TA = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$ and $A^TA = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$ are positive definite;

$A^TA = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ is singular so not positive definite.

6. Section 6.5, Problem 15

Answer: If A and B are positive definite, then $A + B$ is positive definite: Since $\mathbf{x}^{\mathbf{T}}A\mathbf{x} > 0$ and $\mathbf{x}^TB\mathbf{x} > 0$ we have $\mathbf{x}^T(A+B)\mathbf{x} = \mathbf{x}^{\mathbf{T}}A\mathbf{x} + \mathbf{x}^{\mathbf{T}}B\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$. So $A + B$ is positive definite.

7. Section 6.5, Problem 20

Answer:

a) Every positive definite matrix is invertible because the determinant is positive (non-zero).

b) The only positive definite projection matrix is $P = I$ because all projection matrices besides I are singular.

c) A diagonal matrix with positive diagonal entries is positive definite because its eigenvalues are the diagonal entries and they're all positive.

d) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is a symmetric matrix with positve determinant that is not positive definite

8. Section 6.5, Problem 28

Answer:

a) det(A)$= 10$;

b) eigenvalues of A are 2 and 5;

c) eigenvectors are $(cos\theta, sin\theta)$ and $(-sin\theta, cos\theta)$;

d) A is symmetric positive definite because all eigenvalues are positive.