18.06 Spring 2006 - Problem Set 4

SOLUTIONS TO SELECTED PROBLEMS

1. Section 3.6, Problem 15

Answer: If the first two rows of A are exchanged then the **row space** and **nullspace** stay the same.

$\begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ is in the new column space.

2. Section 3.6, Problem 23

Answer:

Basis for row space: $\begin{bmatrix} 3 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$.

Basis for column space: $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$.

A is 3×3 but rank of A is 2, so A is not invertible.

3. Section 3.6, Problem 25

Answer:

a) True; A and A^T have the same rank.

b) False; $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$

c) False; $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

d) True; $\mathbf{C}(A) = \mathbf{C}(-A)$ and $\mathbf{RS}(A) = \mathbf{RS}(-A)$.

4. Section 4.1, Problem 4

Answer: If $AB = 0$ then the columns of B are in the **nullspace** of A. The rows of A are in the **left nullspace** of B.

A and B can't be 3 by 3 matrices of rank 2 because that would mean the four subspaces of A and B all have dimension 2.

5. Section 4.1, Problem 5

Answer:

a) If $A\mathbf{x} = \mathbf{b}$ has a solution and $A^T\mathbf{y} = \mathbf{0}$, then $\mathbf{y}$ is perpendicular to $\mathbf{b}$.

b) If $A^T\mathbf{y} = \mathbf{c}$ has a solution and $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ is perpendicular to $\mathbf{c}$.

6. Section 4.1, Problem 17

Answer:

If $\mathbf{S}$ is the subspace of $\mathbb{R}^3$ containing only the zero vector, then $\mathbf{S}^\perp = \mathbb{R}^3$.

If $\mathbf{S}$ is spanned by $(1, 1, 1)$, $\mathbf{S}^\perp$ is spanned by $(1, -1, 0), (1, 0, -1)$.

If $\mathbf{S}$ is spanned by $(2, 0, 0), (0, 0, 3)$, $\mathbf{S}^\perp$ is spanned by $(0, 1, 0)$.

(Use the basis vectors of $\mathbf{S}$ as rows of a matrix. Then a basis for $\mathbf{S}^\perp$ is the basis of the nullspace.)

7. Section 4.1, Problem 22

Answer: $\mathbf{P}$ is the nullspace of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$. A basis for $\mathbf{P}^\perp$ is $(1, 1, 1, 1)$.

8. Section 4.2, Problem 13

Answer: $P\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

9. Section 4.2, Problem 17

Answer: If $P^2 = P$, then $(I-P)^2 = I^2 - 2P + P^2 = I - P$. When P projects onto the column space of A, $I - P$ projects onto the **left nullspace** of A. ($(I-P)\mathbf{b} = \mathbf{b} - P\mathbf{b} = \mathbf{e}$.)

10. Section 4.2, Problem 19

Answer: For any choice, say $(1,1,0), (2,0,1)$, the matrix P is $\begin{bmatrix} \frac{5}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

11. Section 4.2, Problem 27

Answer: **If $\mathbf{A^T A x = 0}$, then $\mathbf{Ax = 0}$**. The vector $A\mathbf{x}$ is in the nullspace of $\mathbf{A^T}$. $A\mathbf{x}$ is always in the column space of $\mathbf{A}$. To be in both perpendicular spaces, $A\mathbf{x}$ must be zero.

12. COMPUTER PROBLEM:

Answer: For a 5 by 5 matrix, there are $5! = 120$ terms; 60 are positive and 60 are negative. For a 6 by 6 matrix there are $6! = 720$ terms; 360 are positive and 360 are negative.

5 by 5 case in Maple:

```
> with(linalg):
> A:=<<a|b|c|d|e>,<f|g|h|i|j>,<k|l|m|n|o>,<p|q|r|s|t>,<u|v|w|x|y>>;
```

$$A := \begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \\ p & q & r & s & t \\ u & v & w & x & y \end{bmatrix}$$

```
> det(A);
```

$$\begin{aligned} & a\,g\,m\,s\,y - a\,g\,m\,t\,x - a\,g\,r\,n\,y + a\,g\,r\,o\,x + a\,g\,w\,n\,t - a\,g\,w\,o\,s \\ & - a\,l\,h\,s\,y + a\,l\,h\,t\,x + a\,l\,r\,i\,y - a\,l\,r\,j\,x - a\,l\,w\,i\,t + a\,l\,w\,j\,s \\ & + a\,q\,h\,n\,y - a\,q\,h\,o\,x - a\,q\,m\,i\,y + a\,q\,m\,j\,x + a\,q\,w\,i\,o - a\,q\,w\,j\,n \\ & - a\,v\,h\,n\,t + a\,v\,h\,o\,s + a\,v\,m\,i\,t - a\,v\,m\,j\,s - a\,v\,r\,i\,o + a\,v\,r\,j\,n \\ & - f\,b\,m\,s\,y + f\,b\,m\,t\,x + f\,b\,r\,n\,y - f\,b\,r\,o\,x - f\,b\,w\,n\,t + f\,b\,w\,o\,s \\ & + f\,l\,c\,s\,y - f\,l\,c\,t\,x - f\,l\,r\,d\,y + f\,l\,r\,e\,x + f\,l\,w\,d\,t - f\,l\,w\,e\,s \\ & - f\,q\,c\,n\,y + f\,q\,c\,o\,x + f\,q\,m\,d\,y - f\,q\,m\,e\,x - f\,q\,w\,d\,o + f\,q\,w\,e\,n \\ & + f\,v\,c\,n\,t - f\,v\,c\,o\,s - f\,v\,m\,d\,t + f\,v\,m\,e\,s + f\,v\,r\,d\,o - f\,v\,r\,e\,n \\ & + k\,b\,h\,s\,y - k\,b\,h\,t\,x - k\,b\,r\,i\,y + k\,b\,r\,j\,x + k\,b\,w\,i\,t - k\,b\,w\,j\,s \\ & - k\,g\,c\,s\,y + k\,g\,c\,t\,x + k\,g\,r\,d\,y - k\,g\,r\,e\,x - k\,g\,w\,d\,t + k\,g\,w\,e\,s \\ & + k\,q\,c\,i\,y - k\,q\,c\,j\,x - k\,q\,h\,d\,y + k\,q\,h\,e\,x + k\,q\,w\,d\,j - k\,q\,w\,e\,i \end{aligned}$$

- k v c i t + k v c j s + k v h d t - k v h e s - k v r d j + k v r e i

- p b h n y + p b h o x + p b m i y - p b m j x - p b w i o + p b w j n

+ p g c n y - p g c o x - p g m d y + p g m e x + p g w d o - p g w e n

- p l c i y + p l c j x + p l h d y - p l h e x - p l w d j + p l w e i

+ p v c i o - p v c j n - p v h d o + p v h e n + p v m d j - p v m e i

+ u b h n t - u b h o s - u b m i t + u b m j s + u b r i o - u b r j n

- u g c n t + u g c o s + u g m d t - u g m e s - u g r d o + u g r e n

+ u l c i t - u l c j s - u l h d t + u l h e s + u l r d j - u l r e i

- u q c i o + u q c j n + u q h d o - u q h e n - u q m d j + u q m e i