Exam 3, Friday May 4th, 2005

Solutions

Question 1. (a) The characteristic polynomial of the matrix A is

$$-\lambda^3 + \frac{3}{2}\lambda^2 - \frac{9}{16}\lambda + \frac{1}{16} = -(\lambda - 1)\left(\lambda - \frac{1}{4}\right)^2$$

and thus the eigenvalues of A are 1 with multiplicity one and $\frac{1}{4}$ with multiplicity two. Clearly the eigenvectors with eigenvalue 1 are the non-zero multiples of the vector $(1, 1, 1)^T$. The remaining eigenvectors are all non-zero vector of the orthogonal complement of $(1, 1, 1)^T$: they are the vectors

$$\begin{pmatrix} a \\ b \\ -a-b \end{pmatrix}, \quad \text{with } (a, b) \neq (0, 0)$$

and an orthogonal basis for this vector space is

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

(b) For the matrix S we may choose the orthogonal matrix

$$S = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix}$$

and the matrix Λ is then the diagonal matrix with entries 1, $\frac{1}{4}$, and $\frac{1}{4}$ along the diagonal.

We have

$$\lim_{k\to\infty} A^k = S\left(\lim_{k\to\infty} \Lambda^k\right) S^{-1} = S \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} S^T$$

and therefore

$$\lim_{k\to\infty} A^k = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 \end{pmatrix} S^T = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(c) Any $r < \frac{1}{4}$ is such that $A - rI$ is positive definite. Since we want r to be positive, we may choose $r = \frac{1}{8}$.
Any $\frac{1}{4} < s < 1$ is such that $A - sI$ is indefinite. We may choose $s = \frac{1}{2}$.
Any $1 < t$ is such that $A - tI$ is negative definite. We may choose $t = 2$.
(d) The singular values of B are 1, $\frac{1}{2}$ and $\frac{1}{2}$.

Question 2. The trace of A equals the sum of the eigenvalues, which is zero. We deduce that the entry in the second row and second column is $-a$. Similarly, the determinant of A equals the product of the eigenvalues, which is -1. We deduce that the entry in the second row and first column is $1-a^2$. Thus we have

$$A = \begin{pmatrix} a & 1 \\ 1-a^2 & -a \end{pmatrix}$$

(b) The matrix A has two independent eigenvectors since it has two distinct eigenvalues.
(c) The only choices of a giving orthogonal eigenvectors are the ones for which A is symmetric. This implies $a = 0$. If $a \neq 0$, then A does not have orthogonal eigenvectors.
(d) For any choice of a the matrix A has exactly one eigenvalue 1 and exactly one eigenvalue -1. Thus the Jordan canonical form of A is always

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

independently of what a is.

Question 3. (a) The general solution to the differential equation $\frac{du}{dt} = Au$ is

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + c_3 e^{\lambda_3 t} x_3$$

where c_1, c_2, c_3 are arbitrary constants.
(b) Since the vectors x_1, x_2, x_3 are independent, they form a basis for $\mathbb{R}^3$. It follows that we may write any vector $u_0 \in \mathbb{R}^3$ as a linear combination of these vectors: $u_0 = a_1x_1 + a_2x_2 + a_3x_3$. Repeatedly applying the matrix A we obtain

$$u_k = A^k u_0 = \lambda_1^k a_1 x_1 + \lambda_2^k a_2 x_2 + \lambda_3^k a_3 x_3$$

If we want the limit as k goes to infinity of the vectors u_k to be zero, then all the limits $\lim \lambda_i^k$ must be zero. It follows that we necessarily have $-1 < \lambda_i < 1$, for all i's.