18.06 - Spring 2005 - Problem Set 3

Solutions to the Challenge Problems

Problem 1

a) The column space is the space of all vectors whose last $m-r$ coordinates are zero. This is clear since the rank of the matrix R is r and the first r columns of R are independent.

Denote by f_{ij} the entry in the (i,j) position in F. The nullspace of R is the space of all linear combinations of the $n-r$ vectors

$$\begin{pmatrix} -f_{11} \\ -f_{21} \\ \vdots \\ -f_{r1} \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} -f_{12} \\ -f_{22} \\ \vdots \\ -f_{r2} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} -f_{1(n-r)} \\ -f_{2(n-r)} \\ \vdots \\ -f_{r(n-r)} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Clearly these vectors are linearly independent and therefore the dimension of the nullspace is $n-r$.

b) The column space of the matrix B is the same as the column space of R.

Denote by g_{ij} the entry in the (i,j) position in the $r \times (2n-r)$ matrix $G := \begin{pmatrix} F & I & F \end{pmatrix}$. Note that we have $B = \begin{pmatrix} I & G \\ 0 & 0 \end{pmatrix}$. The nullspace of B is the space of all linear combinations of the $2n-r$ vectors

$$\begin{pmatrix} -g_{11} \\ -g_{21} \\ \vdots \\ -g_{r1} \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} -g_{12} \\ -g_{22} \\ \vdots \\ -g_{r2} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} -g_{1(2n-r)} \\ -g_{2(2n-r)} \\ \vdots \\ -g_{r(2n-r)} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

In terms of the matrix F we may write the same vectors as

$$\underbrace{\begin{pmatrix} -f_{11} \\ -f_{21} \\ \vdots \\ -f_{r1} \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} -f_{12} \\ -f_{22} \\ \vdots \\ -f_{r2} \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} -f_{1(n-r)} \\ -f_{2(n-r)} \\ \vdots \\ -f_{r(n-r)} \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{(n-r)\text{ vectors}}, \underbrace{\begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{r\text{ vectors}}, \underbrace{\begin{pmatrix} -f_{11} \\ -f_{21} \\ \vdots \\ -f_{r1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} -f_{1(n-r)} \\ -f_{2(n-r)} \\ \vdots \\ -f_{r(n-r)} \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}}_{(n-r)\text{ vectors}}$$

These vectors are clearly linearly independent, and therefore the nullspace of B has dimension $2n - r$.

c) The column space of C is the space of vectors in $2m-$dimensional space whose coordinates b_i satisfy the equations

$$\begin{aligned} b_i &= b_{i+m} & 1 \le i \le m \\ b_j &= 0 & r+1 \le j \le m \end{aligned}$$

i.e. they are the vectors of the form

$$\begin{pmatrix} b_1 \\ \vdots \\ b_r \\ 0 \\ \vdots \\ 0 \\ b_1 \\ \vdots \\ b_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The nullspace of C is the same as the nullspace of R.

d) The column space of D is the same as the column space of C.

The nullspace of D is the same as the nullspace of B.

Problem 2

a) The nullspace of A is contained in the nullspace of A^2. The reason is that if $Ax = 0$, i.e. if x is in the nullspace of A, then $A^2x = A \cdot \big(Ax\big) = 0$. Thus x is also in the nullspace of A^2. Similarly we have

$$N(A) \subset N(A^2) \subset N(A^3) \subset \dots$$

Note that one can prove that if A is an $n \times n$ matrix, then one has $N(A^n) = N(A^{n+1}) = \dots$.

b) The nullspace is by definition the set of all vectors v such that $\frac{d^2}{dx^2}v = 0$. This means that the polynomial v must be linear: $v = cx + d$. Thus the nullspace is the space of polynomials of degree at most one.

The nullspace of $\left(\frac{d^2}{dx^2}\right)^2$ is the nullspace of the composition of $\frac{d^2}{dx^2}$ with itself: it is the nullspace of $\frac{d^4}{dx^4}$. Thus the nullspace of $\frac{d^4}{dx^4}$ is the space of all polynomials of degree at most three: $v = ax^3 + bx^2 + cx + d$.