18.06 - Spring 2005 - Problem Set 2

Solution to the Challenge Problem

Denote by C_{ij} the $n \times n$ matrix having a 1 in the (i, j) entry, and 0's everywhere else. In terms of these matrices we may write $E_{ij} = I - \ell_{ij} C_{ij}$ and $E_{ij}^{-1} = I + \ell_{ij} C_{ij}$. Thus we have

$$ME_{ij}^{-1} = M\big(I + \ell_{ij} C_{ij}\big) = M + \ell_{ij} MC_{ij}$$

The matrix MC_{ij} has all columns different from the $j-$th consisting entirely of 0's. The $j-$th column of MC_{ij} is simply the $i-$th column of M. Since the matrix E_{ij} is a lower triangular matrix we have $i > j$, and since M is only filled in up to column j, the $i-$th column of M has exactly one 1 in the $i-$th row and 0's everywhere else. Therefore the matrix MC_{ij} has exactly one non-zero entry in position (i, j) and this entry is a 1, i.e. $MC_{ij} = C_{ij}$

We conclude that $ME_{ij}^{-1} = M + \ell_{ij} C_{ij} = N$, as required.

The effect of *right* multiplication by E_{ij}^{-1} on a matrix M is to leave the columns of M different from the $j-$th one unchanged, and replacing the $j-$th column of M by the sum of the $j-$th column of M with ℓ_{ij} times the $i-$th column of M.