18.06 Professor Strang Final Exam May 20, 2004

Your name is: ______________________________

Please circle your recitation:

1)	M2	2-131	I. Ben-Yaacov	2-101	3-3299	`pezz`
2)	M3	2-131	I. Ben-Yaacov	2-101	3-3299	`pezz`
3)	M3	2-132	A. Oblomkov	2-092	3-6228	`oblomkov`
4)	T11	2-132	A. Oblomkov	2-092	3-6228	`oblomkov`
5)	T12	2-132	I. Pak	2-390	3-4390	`pak`
6)	T1	2-131	B. Santoro	2-085	2-1192	`bsantoro`
7)	T1	2-132	I. Pak	2-390	3-4390	`pak`
8)	T2	2-132	B. Santoro	2-085	2-1192	`bsantoro`
9)	T2	2-131	J. Santos	2-180	3-4350	`jsantos`

Problems 1–8 are 12 points each; Problem 9 is 4 points.

Thank you for taking 18.06!

1 Suppose A is an m by n matrix of rank r. You multiply it by any m by n invertible matrix E to get $B = EA$.

(a) Circle if true and cross out if false (three parts):

$$A \text{ and } B \text{ have the } \begin{cases} \textit{same nullspace} \\ \textit{same column space} \\ \textit{same bases for row space.} \end{cases}$$

(b) Suppose the right E gives the row-reduced echelon matrix

$$EA = R = \begin{bmatrix} 1 & 4 & 0 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(**1**) Find a basis for the nullspace of A.

(**2**) True statement: *The nullspace of a matrix is a vector space.*

What does it mean for a set of vectors to be a vector space?

(c) What is the nullspace of a 5 by 4 matrix with linearly independent columns?

What is the nullspace of a 4 by 5 matrix with linearly independent columns?

2 This matrix A has column 1 + column 2 = column 3:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) Describe the column space $\boldsymbol{C}(A)$ in two ways:

(**1**) Give a basis for $\boldsymbol{C}(A)$.

(**2**) Find all vectors that are *perpendicular* to $\boldsymbol{C}(A)$.

(b) The projection matrix P onto the column space does not come from the usual formula $A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. *Why not*—what goes wrong with this formula?

(c) Find that matrix P for projection onto the column space of A.

3 Suppose P is the 3 by 3 projection matrix (so $P = P^{\mathrm{T}} = P^2$) onto the plane $2x + 2y - z = 0$. You do not have to compute this matrix P but you can if you want.

(a) What is the rank of P? What are its three eigenvalues? What is its column space?

(b) Is P diagonalizable—why or why not? Find a nonzero vector in its nullspace.

(c) If b is any unit vector in $\mathbf{R}^3$, find the number q. *Explain your thinking in 1 sentence and 1 equation*:

$$q = \|Pb\|^2 + \|b - Pb\|^2 \,.$$

4 **(a)** If $a \neq c$, find the eigenvalue matrix Λ and eigenvector matrix S in

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = S \Lambda S^{-1}.$$

(b) Find the *four entries* in the matrix A^{1000}.

5 **(a)** Suppose $A^{\mathrm{T}}Ax = 0$. This tells us that Ax is in the ____ space of A^{T}. Always Ax is in the ____ space of A. Why can you conclude that $Ax = 0$?

(b) Supposing again that $A^{\mathrm{T}}Ax = 0$ we immediately get $x^{\mathrm{T}}A^{\mathrm{T}}Ax = 0$.

From this, *show directly that* $Ax = 0$.

Every matrix $A^{\mathrm{T}}A$ is symmetric and ____________.

(c) The rectangular m by n matrix A always has the same nullspace as the square matrix $A^{\mathrm{T}}A$ (this is proved above). Now deduce that A and $A^{\mathrm{T}}A$ have the *same rank*.

6 Suppose $A = \mathsf{ones}(3,5)$ and $A^{\mathrm{T}} = \mathsf{ones}(5,3)$ are the 3 by 5 and 5 by 3 matrices of all 1's.

(a) Find the trace of AA^{T} and the trace of $A^{\mathrm{T}}A$.

(b) Find the eigenvalues of AA^{T} and the eigenvalues of $A^{\mathrm{T}}A$.

(c) What is the matrix Σ in the singular value decomposition $A = U\Sigma V^{\mathrm{T}}$?

7 **(a)** By elimination or otherwise, *find the determinant of* A:

$$A = \begin{bmatrix} 1 & 0 & 0 & u_1 \\ 0 & 1 & 0 & u_2 \\ 0 & 0 & 1 & u_3 \\ v_1 & v_2 & v_3 & 0 \end{bmatrix}$$

(b) If that *zero* in the lower right corner of A changes to 100, what is the change (if any) in the determinant of A? (You can consider its cofactors)

(c) If (u_1, u_2, u_3) is the same as (v_1, v_2, v_3) so A is symmetric, decide if A is or is not positive definite—and why?

(d) Show that this block matrix M is singular for any u and v in $\mathbf{R}^n$, by finding a vector in its nullspace:

$$M = \begin{bmatrix} I & u \\ v^{\mathrm{T}} & v^{\mathrm{T}}u \end{bmatrix}.$$

8 Suppose q_1, q_2, q_3 are orthonormal vectors in $\mathbf{R}^4$ (not $\mathbf{R}^3$!).

(a) What is the length of the vector $v = 2q_1 - 3q_2 + 2q_3$?

(b) What four vectors does Gram-Schmidt produce when it orthonormalizes the vectors q_1, q_2, q_3, u?

(c) If u in part (b) is the vector v in part (a), why does Gram-Schmidt break down? Find a nonzero vector in the nullspace of the 4 by 4 matrix

$$A = \begin{bmatrix} q_1 & q_2 & q_3 & v \end{bmatrix} \quad \text{with columns } q_1, q_2, q_3, v\,.$$

9 **(4 points)** PROVE (give a clear reason): If A is a symmetric invertible matrix then A^{-1} is also symmetric.