18.06 Professor A.J. de Jong Exam 2 April 9, 2003

Your name is: ______________________________

Please circle your recitation:

Important: Briefly explain all of your answers.

1 (29 pts.)

(a) Compute the determinant of the following matrix

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = -1$$

We expanded the determinant along row 1 then subtracted row 1 from rows 3 and 4 and then expanded the determinant along the 1st column. The last 3x3 determinant was computed directly.

(b) Give a basis for each of the four fundamental subspaces associated to the following matrix

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We switched rows 1 and 2 then subtracted row 1 from row 3 and then subtracted row 2 from row 3.

First two rows (0, 1, -1 , 0) and (1, 0, -1, 0) is a basis for the row space.

First two columns (0, 1, 1) and (1, 0, -1) is a basis for the column space.

Solving $Av = 0$ we get $x_1 = x_2 = x_3$. Thus,(1, 1, 1, 0) and (0, 0, 0, 1) is a basis for the $Null(A)$ space.

Solving $A^t v = 0$ we get $x_1 = -x_2 = x_3$. Thus, (1, -1, 1) is a basis for the $Null(A^t)$ space.

2 (29 pts.)

(a) Apply the Gram-schmidt algorithm to the columns of the matrix A below. (Use the order in which they occur in the matrix!) Use this to write $A = QR$, where Q is a matrix with orthonormal columns, and R is upper triangular.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & -1 \end{pmatrix}.$$

$q_1 = a_1 = (1, 0, 0, -1)$.
$q_2 = a_2 - \frac{(a_2 \cdot q_1)}{(q_1 \cdot q_1)} q_1 = (0, 1, 0, -1) - (1/2, 0, 0, -1/2) = (-1/2, 1, 0, -1/2)$.
Normalize $q_1 = (1/\sqrt{2}, 0, 0, -1/\sqrt{2})$, $q_2 = (-1/\sqrt{6}, 2/\sqrt{6}, 0, -1/\sqrt{6})$. $Q = [q_1, q_2]$, $R = Q^t A$.

$$A = QR = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \\ 0 & 0 \\ -1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} \end{pmatrix}.$$

(b) Compute the matrix of the projection onto the column space of A. What is the distance of the point $(1, 1, 1, 0)$ to this column space?

$$P = QQ^t = \begin{pmatrix} 2/3 & -1/3 & 0 & -1/3 \\ -1/3 & 2/3 & 0 & -1/3 \\ 0 & 0 & 0 & 0 \\ -1/3 & -1/3 & 0 & 2/3 \end{pmatrix}.$$

If $b = (1, 1, 1, 0)$ then its projection is $p = Pb = (1/3, 1/3, 0, -2/3)$. The distance $d = \|b - p\| = \|(2/3, 2/3, 1, 2/3)\| = \sqrt{21}/3$.

3 (14 pts.) Show that the following determinant is zero for any values of a, b and c:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a+b+c & b+c+a & c+a+b \end{vmatrix} = 0$$

We added row 2 to row 3. The determinant is 0 since rows 1 and 3 are multiples of each other.

4 (28 pts.) Let A be the matrix

$$\begin{pmatrix} 7 & 5 \\ 3 & -7 \end{pmatrix}$$

(a) Find matrices S and Λ such that A has a factorization of the form

$$A = S\Lambda S^{-1},$$

where S is invertible and Λ is diagonal: $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$.

$det(A - \lambda I) = 0$. The eigenvalues: $\lambda^2 - 64 = 0$, $\lambda_1 = 8$, $\lambda_2 = -8$. Eigenvectors:

$(A - \lambda_1 I)v_1 = \begin{pmatrix} -1 & 5 \\ 3 & -15 \end{pmatrix} v_1 = 0$ then $v_1 = (5, 1)$,

$(A - \lambda_2 I)v_2 = \begin{pmatrix} 15 & 5 \\ 3 & 1 \end{pmatrix} v_2 = 0$ then $v_2 = (1, -3)$.

$S = \begin{pmatrix} 5 & 1 \\ 1 & -3 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 8 & 0 \\ 0 & -8 \end{pmatrix}$, $S^{-1} = (-1/16)\begin{pmatrix} -3 & -1 \\ -1 & 5 \end{pmatrix}$.

(b) Find a matrix B such that $B^3 = A$. (Hint: First find such a matrix for Λ. Then use the formula above.)

$B = S\Lambda^{1/3}S^{-1} = \begin{pmatrix} 5 & 1 \\ 1 & -3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \cdot (-1/16)\begin{pmatrix} -3 & -1 \\ -1 & 5 \end{pmatrix} = 1/4\begin{pmatrix} 7 & 5 \\ 3 & -7 \end{pmatrix}$.