## 18.06 Problem Set #7 Solutions

1. First we want to find eigenvalues and eigenvectors of $A$. The characteristic equation for $A$ is

$$0 = \det(\lambda I - A) = \lambda^3 - 12\lambda - 16 = (\lambda + 2)^2(\lambda - 4)$$

So we have eigenvalues of $A$ as -2, -2 and 4. The eigenvectors corresponding to -2 are (-1,1,0) and (-1,0,1), an eigenvector for 4 is (1,1,1). You can have equivalent set of eigenvectors.

Secondly we want to apply Gram-Schmidt method to find eigenvectors which also form a set of orthonormal basis. We apply Gram-Schmidt to the set of eigenvectors corresponding to the same eigenvalue for every eigenvalue. The reason is that we know eigenvectors corresponding to different eigenvalues are orthogonal. For the current problem, we need to do Gram-Schmidt for $\{(-1,1,0),(-1,0,1)\}$ and get $\{\frac{1}{\sqrt{2}}(-1,1,0), \frac{1}{\sqrt{6}}(-1,-1,2)\}$. For (1,1,1) we get $\frac{1}{\sqrt{3}}(1,1,1)$. The final answer to the problem is (the answer to Q is not unique)

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \Sigma = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, A = Q\Sigma Q^T$$

2. The numbers in the problem don't have a neat computation, we will round off the work. The final answer will be approximate orthogonal matrices ($Q^TQ$ is close to identity matrix). The singular values we get will also be approximate singular values.

First we find eigenvalues of $B^TB = \begin{bmatrix} 5 & -13 \\ -13 & 37 \end{bmatrix}$, which are 41.6 and 0.4. The corresponding eigenvectors are (1, -2.8) and (2.8, 1) respectively. Multiply each eigenvector by reciprocal of its norm and get unit eigenvectors (0.33, -0.94) and (0.94, 0.33).

If $B = V\Sigma U^T$ for $U$, $V$ orthogonal and $\Sigma$ diagonal, then $U = \begin{bmatrix} 0.33 & 0.94 \\ -0.94 & 0.33 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{41.6} & 0 \\ 0 & \sqrt{0.4} \end{bmatrix} = \begin{bmatrix} 6.5 & 0 \\ 0 & 0.62 \end{bmatrix}$, $V = AU\Sigma^{-1} = \begin{bmatrix} 0.2 & 0.98 \\ -0.98 & 0.2 \end{bmatrix}$.

3. By linearity, $L(6,2,2,6) = L(6(1,0,0,1) + 2(0,1,1,0)) = 6(2,3) + 2(1,5) = (14,28)$.

4. Still use $S, T, S', T'$ to denote the basis matrix of each set of basis, i.e.

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, S' = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, T' = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Then the matrices for $L$ in each case are

(a) $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $T'^{-1}AS' = \frac{1}{3}\begin{bmatrix} 7 & -4 \\ -2 & 5 \\ 2 & -2 \end{bmatrix}$

(c) $T'^{-1}AS = \frac{1}{3}\begin{bmatrix} 3 & -4 \\ 3 & 5 \\ 0 & -2 \end{bmatrix}$ (d) $T^{-1}AS' = \begin{bmatrix} 3 & -2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$

5. If consider $S$ to be a standard basis for $M_2$, basis vectors in the given order, then basis matrix of $T$ is (also denoted by $T$)

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Basis matrix for $S$, also denoted by $S$, is the identity 4x4 matrix. Now similar to problem 4, matrices for the linear transform $L$ are

(a) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $T^{-1}AT = \begin{bmatrix} 2 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 \end{bmatrix}$

(c) $T^{-1}AS = \begin{bmatrix} 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

(d) $S^{-1}AT = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$