18.06 Final Exam (Conflict Exam), Spring, 2001

Name ____________________ Optional Code ____________________

Recitation Instructor ____________________ Email Address ____________________

Recitation Time ____________________

This final exam is closed book and closed notes. No calculators, laptops, cell phones or other electronic devices may be used during the exam.

There are 6 problems.

Additional paper for your calculations is provided at the back of this booklet.

Good luck.

Problem	Maximum Points	Your Points
1.	15	
2.	15	
3.	15	
4.	20	
5.	20	
6.	15	
Total	100	

1. (15pts.) For which values of a and b does the system of equations

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & ax_3 & + & 2x_4 & = & 1 \\ x_1 & + & & + & 3x_3 & + & 4x_4 & = & b \\ 2x_1 & + & x_2 & + & (a+b)x_3 & + & 7x_4 & = & 2 \end{array}$$

have **no** solutions? Find all solutions in the case that $a = 7$ and $b = 1$.

Additional paper for your calculations at the back of this booklet.

2. (15pts.) Let A_n be the $n \times n$ matrix

$$A_n = \begin{pmatrix} 1 & 2 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 2 & \cdots & 0 & 0 \\ 0 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 2 & 1 \end{pmatrix}.$$

Prove that for $n \geq 3$, $\det(A_n) = \det(A_{n-1}) - 4\det(A_{n-2})$, and evaluate $\det(A_5)$.

Additional paper for your calculations at the back of this booklet.

3. (15pts.) The following are some quick questions. Give only brief reasoning for your answers, no detailed proofs.

(a) Let A, B, C and D be four 3×3 matrices. Let E be the 6×6 matrix

$$E = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Is it necessarily true that $\det(E) = \det(A) \cdot \det C - \det B \cdot \det D$?

(b) Let A be a 3×4 matrix, and B be a 4×3 matrix. Can you say anything about the determinant of their product, BA? How about AB?

(c) Do similar matrices have the same

i. eigenvalues;

ii. eigenvectors;

iii. rank;

iv. column space;

v. determinant?

(d) Does an $n \times n$ matrix with n distinct eigenvalues have an orthogonal set of eigenvectors?

(e) Is the product of two symmetric matrices symmetric?

Additional paper for your calculations at the back of this booklet.

4. (20pts.) Let V be the vector space of polynomials of degree at most 3 with real coefficients. Let T be the map defined by

$$T(f(x)) = f(x) - (1+x)\frac{df}{dx}$$

for all $f(x) \in V$.

(a) Show that T is a linear transformation.

(b) Find the matrices ${}_B(T)_B$ and ${}_C(T)_C$ representing T with respect to the bases $B = \{1, x, x^2, x^3\}$ and $C = \{1+x, x+x^2, x^2+x^3, x^3\}$.

(c) Find the matrix ${}_C(I)_B$ representing the change of basis from B to C, and verify that ${}_C(T)_C = {}_C(I)_{BB}(T)_{BB}(I)_C$.

(d) Find bases for the kernel and image of T.

Additional paper for your calculations at the back of this booklet.

5. (20pts.) Let U and V be vector spaces.

(a) Define the *kernel* and *image* of a linear transformation $T : U \to V$.

(b) Show that the kernel of T is a subspace of U.

(c) Let T be a linear transformation from U to V and let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ form a basis of Ker T. The following steps help you to show that if $\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n$ form a basis of U, then $T\mathbf{u}_{k+1}, \ldots, T\mathbf{u}_n$ form a basis of Im T. So assume that $\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1} \ldots, \mathbf{u}_n$ are a basis of U.

i. Argue that $T\mathbf{u}_{k+1}, \ldots, T\mathbf{u}_n$ are elements of Im T.

ii. Show that any element of Im T can be expressed as a linear combination of $T\mathbf{u}_{k+1}, \ldots, T\mathbf{u}_n$.

iii. Show that $T\mathbf{u}_{k+1}, \ldots, T\mathbf{u}_n$ are linearly independent.

(d) Deduce a formula relating the dimensions of U, Ker T and Im T.

Additional paper for your calculations at the back of this booklet.

6. (15pts.) Find the singular value decomposition of the 3×2 matrix

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \\ 1 & 2 \end{pmatrix}.$$

Additional paper for your calculations at the back of this booklet.

Your calculations for problem ____.

Your calculations for problem ____.

Your calculations for problem ____.

Your calculations for problem ____.