18.06 Exam 2 Solutions April 25, 2000

1 (30 pts.) (a) $q_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, q_2 = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, q_3 = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$

Find q_3 by finding the left nullspace of A then normalising, or by taking $q_1 \times q_2$, or by guessing a vector independent of q_1 and q_2 and using Gram Schmidt.

(b) q_3 is in the left nullspace of A, since it is orthogonal to both columns of A

(c) $P = q_3(q_3^T q_3)^{-1} q_3^T = \frac{1}{9}\begin{bmatrix} 4 & -4 & -2 \\ -4 & 4 & 2 \\ -2 & 2 & 1 \end{bmatrix}$

(d) $\hat{x} = (A^T A)^{-1} A^T \vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

2 (16 pts.) Big formula: det of $A = 16 - 4 - 4 - 4 + 1 = 5$

$$\text{Row reduce: det of } A = \text{det of} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix} = 5$$

3 (30 pts.) (a) $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$

(b) $\lambda_1 = 1$ with e-vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -\frac{1}{2}$ with e-vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(c) $A^k = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1^k & 0 \\ 0 & (-\frac{1}{2})^k \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$

(d) $A^\infty = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$

$$\begin{pmatrix} G_\infty \\ G_\infty \end{pmatrix} = A^\infty \begin{pmatrix} G_1 \\ G_0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

So $G_k \to \frac{2}{3}$

4 (24 pts.)

(a) $n - r =$ dimension of the nullspace of $A =$ the number of e-vals of A which are 0. So $r = 2$

(b) $det(A^T A) = det(A^T)det(A) = det(A)det(A)$,
and $det(A) = 0 * 1 * 2 = 0$, so $det(A^T A) = 0$

(c) When we add I to a matrix, it increases the e-vals by 1. So the e-vals of $A + I$ are $1, 2, 3$, and $det(A + I) = 1 * 2 * 3 = 6$

(d) If A has e-val λ, then A^{-1} has e-val $\frac{1}{\lambda}$. So the e-vals of $(A + I)^{-1}$ are $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}$