18.06 Quiz 3 December 3, 1999 Closed Book

1 (36 pts.) The differential equation is

$$\frac{du}{dt} = Au \qquad \text{with} \qquad A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \qquad \text{and} \qquad u(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors and diagonalize to $A = S\Lambda S^{-1}$.

A is not invertible, hence one eigenvalue is 0.

$Tr(A) = -5$, so the other eigenvalue of A must be -5.

An eigenvector of A with eigenvalue 0 is $(3, 2)$.

An eigenvector of A with eigenvalue -5 is $(1, -1)$.

$$A = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{bmatrix}.$$

(b) Solve for $u(t)$ starting from the given $u(0)$.

$$\text{General solution is} \quad u(t) = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \, e^{-5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\text{The condition} \quad u(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \quad \text{is satisfied when} \quad c_1 = 1, \; c_2 = 2.$$

(c) Compute the matrix e^{At} using S and Λ.

$A = S\Lambda S^{-1} \Rightarrow e^{At} = Se^{\Lambda t}S^{-1}$. So

$$e^{At} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-5} \end{bmatrix} \begin{bmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2e^{-5t} + 3 & -3e^{-5t} + 3 \\ -2e^{-5t} + 2 & 3e^{-5t} + 2 \end{bmatrix}$$

(d) As t approaches infinity, find the limits of $u(t)$ and e^{At}.

$$\text{As} \quad t \to \infty, \quad e^{-5t} \to 0, \quad u(t) \to \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \text{and} \quad e^{At} \to \frac{1}{5} \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$$

2 (40 pts.) The matrix A has 3's on the diagonal and 2's everywhere else:

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

(a) Decide if A is positive definite. What is the minimum value of $x^T Ax$ for **all** vectors x in $\mathbb{R}^3$?

A is positive definite since:

1. A is symmetric,
2. The upper left determinants are $3, 5, 7$.

A is positive definite $\Rightarrow x^T Ax \geq 0$.

When $x = 0$, $x^T Ax = 0$. So 0 is the minimum.

(b) All entries of $B = A - I$ are 2's. From its rank find all the eigenvalues of B and then all the eigenvalues of A.

All three columns of B are the same

$\Rightarrow Rank(B) = 1$

$\Rightarrow N(B)$ has dimension 2

$\Rightarrow B$ has two independent eigenvectors with eigenvalue 0

$\Rightarrow \lambda_1 = \lambda_2 = 0$, and $\lambda_3 = tr(B) = 6$.

The eigenvalues of A are $1, 1, 7$.

(c) Write down any one specific symmetric matrix C that is similar to A. Write down if possible any one nonsymmetric matrix N that is similar to A. Write down a matrix J with the same eigenvalues as A that is **not** similar to A. (Give the 9 numbers in C, N, J.)

$$C = \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

Explanation: A is symmetric(so you could have let $C = A$) and hence can be diagonalized to Λ. Consider

$$D = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 7 \end{bmatrix}$$

D has eigenvalues 1, 1, 7.

If D is similar to A, then D can also be diagonalized to Λ and hence must have two eigenvectors with eigenvalue 1. This is possible iff $x = 0$. Our choice of C is obtained by letting $y = z = 0$. Our choice of N is obtained by letting $y = 0$, $z = 1$.

When $x \neq 0$, D is not similar to A. Our choice of J is obtained by letting $x = 1$.

Note: There are choices for C, N, and J which are not upper triangular.

(d) For the 6 by 6 matrix A_6 with 3's on the diagonal and 2's everywhere else use the same method (with $A_6 - I$) to find the six eigenvalues. If you make a good choice of eigenvectors, in what form can you factor A?

The matrix $B_6 = A_6 - I$ has rank 1, so it has 5 independent eigenvectors with eigenvalue 0. It follows that
the eigenvalues of B_6 are $0, 0, 0, 0, 0, 12 = tr(B)$ and
the eigenvalues of A_6 are $1, 1, 1, 1, 1, 13$.

We already know that $A = S\Lambda S^{-1}$ for some S whose columns are independent eigenvectors of A. But A is symmetric, so we can choose its eigenvectors to be orthonormal and have $A = Q\Lambda Q^{-1}$, where Q is orthogonal.

3 (24 pts.) Suppose $A = U\Sigma V^T =$ (orthogonal 2×2) (diagonal) (orthogonal 3×3)

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

(a) What are the eigenvalues and eigenvectors of $A^T A$?

$A^T A = (U\Sigma V^T)^T (U\Sigma V^T) = V\Sigma^T \Sigma V^T$.

$$\Sigma^T \Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of $A^T A$ are $16, 1, 0$.

v_1 is an eigenvector of $A^T A$ with eigenvalue 16.

v_2 is an eigenvector of $A^T A$ with eigenvalue 1.

v_3 is an eigenvector of $A^T A$ with eigenvalue 0.

(b) What is the nullspace of A? (Describe the whole nullspace.)

The nullspace of A is the linear span of v_3.

(c) What is the row space of A? (Describe the whole row space.)

The row space of A is the linear span of v_1 , v_2.