**18.06** Quiz 2 Solution November 10, 1999 Closed Book

**1 (25 pts.)** (a) Find equations (**do not solve**) for the coefficients $C, D, E$ in $b = C + Dt + Et^2$, the parabola which best fits the four points $(t, b) = (0,0), (1,1), (1,3)$ and $(2,2)$.

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$ has no solution. We need to look for its least square solution and solve the system

$$A^T A \begin{bmatrix} C \\ D \\ E \end{bmatrix} = A^T \mathbf{b}, \qquad \text{i.e.,} \qquad \begin{bmatrix} 4 & 4 & 6 \\ 4 & 6 & 10 \\ 6 & 10 & 18 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}$$

(b) In solving this problem you are projecting the vector $\underline{\mathbf{b} = (0, 1, 3, 2)}$ onto the subspace spanned by <u>the column vectors of $A$</u>. The projection in terms of $C, D, E$ is

$$P = A\hat{x} = A \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} C \\ C + D + E \\ C + D + E \\ C + 2D + 4E \end{bmatrix}$$

**2 (28 pts.)** Let

$$A = \begin{bmatrix} 3 & 4 & 6 \\ 0 & 1 & 0 \\ -1 & -2 & -2 \end{bmatrix}.$$

(a) Find the eigenvalues of the singular matrix $A$.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 4 & 6 \\ 0 & 1-\lambda & 0 \\ -1 & -2 & -2-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)\lambda,$$

so the eigenvalues of $A$ are $0, 1, 1$.

(b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of $A$.

$$A\mathbf{x} = \begin{bmatrix} 3 & 4 & 6 \\ 0 & 1 & 0 \\ -1 & -2 & -2 \end{bmatrix} \mathbf{x} = \mathbf{0} \quad \text{has special solution} \quad \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$(A - I)\mathbf{x} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \\ -1 & -2 & -3 \end{bmatrix} \mathbf{x} = \mathbf{0} \quad \text{has special solutions} \quad \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

So one such basis is

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

(c) By expressing $(1,1,1)$ as a combination of eigenvectors or by diagonalizing $A = S\Lambda S^{-1}$, compute

$$A^{99} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

First method:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 6 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - 5 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

So

$$A^{99} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = A^{99}(6v_1) + A^{99}(v_2) + A^{99}(-5v_3) = 0 + v_2 - 5v_3 = \begin{bmatrix} 13 \\ 1 \\ -5 \end{bmatrix}$$

Second method:

$$A = S \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} S^{-1},$$

$$A^{99} = S \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{99} S^{-1} = S \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} S^{-1} = A.$$

So

$$A^{99} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \\ -5 \end{bmatrix}$$

**3 (25 pts.)** Start with two vectors (the columns of $A$):

$$a_1 = \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix} \quad \text{and} \quad a_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) With $q_1 = a_1$ find an orthonormal basis $q_1, q_2$ for the space spanned by $a_1$ and $a_2$ (column space of $A$).

$$b_2 = a_2 - a_2 \cdot q_1\, q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \cos\theta \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix} = \begin{bmatrix} 1-\cos^2\theta \\ 0 \\ -\cos\theta\sin\theta \end{bmatrix},$$

$$q_2 = \frac{b_2}{|b_2|} = \begin{bmatrix} \sin\theta \\ 0 \\ -\cos\theta \end{bmatrix}.$$

(b) What shape is the matrix $R$ in $A = QR$ and why is $R = Q^T A$? Here $Q$ has columns $q_1$ and $q_2$. Compute the matrix $R$.

$R$ is a $2 \times 2$ upper triangular matrix.

$$A = QR \quad \Rightarrow \quad Q^T A = Q^T QR \quad \Rightarrow \quad Q^T A = IR = R$$

$$R = \begin{bmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{bmatrix}$$

(c) Find the projection matrices $P_A$ and $P_Q$ onto the column spaces of $A$ and $Q$.

$$\text{Since } C(A) = C(Q), \quad P_A = P_Q = QQ^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If you notice that the second entry of both $a_1$ and $a_2$ are zero, then you know you are looking for the projection matrix onto the $xz$-plane. You can obtain the answer without doing any matrix multiplication.

**4 (22 pts.)** (a) If $Q$ is an orthogonal matrix (square with orthonormal columns), show that $\det Q = 1$ or $-1$.

$$\begin{aligned} & Q^TQ = I \\ \Rightarrow \quad & |Q^TQ| = |I| \\ \Rightarrow \quad & |Q^T||Q| = 1 \\ \Rightarrow \quad & |Q||Q| = 1 \quad \text{because } |A^T| = |A| \\ \Rightarrow \quad & |Q| = \pm 1. \end{aligned}$$

(b) How many of the 24 terms in $\det A$ are nonzero, and what is $\det A$?

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

There are four nonzero terms in $\det A$:

$$\begin{bmatrix} \mathbf{1} & 0 & 1 & 0 \\ 0 & \mathbf{1} & 0 & 1 \\ 1 & 0 & \mathbf{-1} & 0 \\ 0 & -1 & 0 & \mathbf{1} \end{bmatrix}, \begin{bmatrix} \mathbf{1} & 0 & 1 & 0 \\ 0 & 1 & 0 & \mathbf{1} \\ 1 & 0 & \mathbf{-1} & 0 \\ 0 & \mathbf{-1} & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 1 \\ \mathbf{1} & 0 & -1 & 0 \\ 0 & -1 & 0 & \mathbf{1} \end{bmatrix}, \begin{bmatrix} 1 & 0 & \mathbf{1} & 0 \\ 0 & 1 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & -1 & 0 \\ 0 & \mathbf{-1} & 0 & 1 \end{bmatrix}$$

Each of the four terms is equal to -1, so det A = -4.