18.06 Final Exam December 16, 1999 Closed Book

1 (12 pts.) Let

$$A = \begin{bmatrix} 7 & 0 & 2 & 4 \\ 7 & 1 & 3 & 6 \\ 14 & -1 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find bases for the four fundamental subspaces.

(b) Find the conditions on b_1, b_2, and b_3 so that

$$Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

has a solution.

(c) If $Ax = b$ has a solution x_p, describe all of the solutions.

2 (10 pts.) Let A and B be any two matrices so that the product AB is defined.

(a) Explain why every column of AB is in the column space of A.

(b) How does part (a) lead to the conclusion that the rank of AB is less than or equal to the rank of A? State your reasoning in logical steps.

3 (10 pts.) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation satisfying

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -10 \\ 8 \end{bmatrix}.$$

(a) Find $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

(b) What is the matrix A expressing T in terms of the standard basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$? (The same basis is used for the input and the output.)

(c) What is the matrix B expressing T in terms of the basis consisting of eigenvectors of A? (The same basis is used for the input and output.) (There are two possible correct answers, depending on what order you pick the eigenvectors.)

4 (16 pts.) Let V be the subspace of $\mathbf{R}^3$ consisting of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying

$$x + 2y - 5z = 0.$$

(a) Find a 3×2 matrix A whose column space is V.

(b) Find an orthonormal basis for V.

(c) Find the projection matrix P projecting onto the left nullspace (not the column space!) of A.

(d) Find the least squares solution to

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

5 (15 pts.) Suppose

$$Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{has no solution}$$

but

$$Ax = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \text{has infinitely many solutions.}$$

(a) Find all possible information about r, m, and n. (The rank and the shape of A.)

(b) Find an example of such a matrix A with r, m, and n all as small as possible.

(c) How do you know that $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is not in the nullspace of A^{T}?

6 (13 pts.) In each case give all the information you can about the eigenvalues and eigenvectors, when the matrix A has the following property:

(a) The powers A^k approach the zero matrix.

(b) The matrix is symmetric positive definite.

(c) The matrix is not diagonalizable.

(d) The matrix has the form $A = uv^{\mathrm{T}}$, where u and v are vectors in $\mathbf{R}^3$. (You might want to try an example.)

(e) A is similar to a diagonal matrix with diagonal entries 1, 1, and 2.

7 (12 pts.) Define a sequence of numbers in the following way: $G_0 = 0$, $G_1 = 1/2$, and $G_{k+2} = (G_{k+1} + G_k)/2$. (Each number is the average of the two previous numbers.)

(a) Set up a 2×2 matrix A to get from $\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$ to $\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix}$.

(b) Find an explicit formula for G_k.

(c) What is the limit of G_k as $k \to \infty$?

8 (12 pts.) (a) Suppose A is a 4×4 matrix of rank 3, and let

$$x = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \\ C_{14} \end{bmatrix}$$

be the cofactors of its first row. Explain why $Ax = 0$. (So the cofactors give a formula for a nullspace vector!)

Hint: The first component of Ax and the second component of Ax are determinants of (different) matrices. What are these matrices and why do they have zero determinants? (The 3rd and 4th components of Ax follow similarly, so you can just answer for the 1st and 2nd components.)

(b) Compute the determinant of

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Hint: You might find it convenient to use the fact that the columns are orthogonal.