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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Hour Exam I for Course 18.06: Linear Algebra

Recitation Instructor:

Your Name: **SOLUTIONS**

Recitation Time:

Lecturer:

Grading

1. **34**
2. **16**
3. **34**
4. **16**

TOTAL: 100

Do all your work on these pages.
No calculators or notes.
Please work carefully, and check your intermediate results whenever possible.
Point values (total of 100) are marked on the left margin.

1. Let $A = \begin{bmatrix} 2 & 1 & 0 & 4 \\ 2 & 1 & 1 & 2 \\ 4 & 2 & 3 & 2 \end{bmatrix}$.

[12] **1a.** Give the LU-factorization of A.

$$A = \begin{bmatrix} 2 & 1 & 0 & 4 \\ 2 & 1 & 1 & 2 \\ 4 & 2 & 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{2} & 1 & 0 & 4 \\ 0 & 0 & \boxed{1} & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

l_{21}: row <u>2</u> $-$ (1) $\times$ row <u>1</u>.
l_{31}: row <u>3</u> $-$ (2) $\times$ row <u>1</u>.
l_{32}: row <u>3</u> $-$ (3) $\times$ row <u>2</u>.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}.$$

[6] **1b.** Give a basis for the column space of A.

Pivot columns in A: column 1, column 3.

$\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ form a basis for the column space of A.

[16] **1c.** For what value of q is $Ax = \underbrace{\begin{bmatrix} 2 \\ 3 \\ q \end{bmatrix}}_{b}$ solvable?

$$\underbrace{\begin{bmatrix} 2 & 1 & 0 & 4 & | & 2 \\ 2 & 1 & 1 & 2 & | & 3 \\ 4 & 2 & 3 & 2 & | & q \end{bmatrix}}_{[A \ | \ b]} \to \begin{bmatrix} 2 & 1 & 0 & 4 & | & 2 \\ 0 & 0 & 1 & -2 & | & 1 \\ 0 & 0 & 3 & -6 & | & q-4 \end{bmatrix} \to \underbrace{\begin{bmatrix} \boxed{2} & 1 & 0 & 4 & | & 2 \\ 0 & 0 & \boxed{1} & -2 & | & 1 \\ 0 & 0 & 0 & 0 & | & q-7 \end{bmatrix}}_{[U \ | \ c]}.$$

The last row of the reduced system $Ux = c$ produces
$0x_1 + 0x_2 + 0x_3 + 0x_4 = (q-7)$.
The reduced system (and the original system $Ax = b$)
are solvable if and only if $\boxed{\text{q=7}}$.

Notice that $b = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$ is the sum of the pivot columns in A.

[8] **2a.** Let $s = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $t = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Suppose we have a matrix A (4 rows, 4 columns) such that $Ax = s$ has no solution and $Ax = t$ has no solution.

The rank of A is at most (circle a number): 1 2 $\boxed{3}$ 4.

`The rank of` A `is at most` 3.

`Example:` $A = \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ `spans an` $r = 3$ `dimensional subspace in` R^4

`and` s, t `are not in the column space of` A.

[8] **2b.** Let $u = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix}$.

Suppose we have a matrix B such that $Bx = u$ has no solution and $Bx = v$ has no solution. Is it also possible that $Bx = u + v$ has infinitely many solutions?
If 'yes', give a matrix B that satisfies these conditions.
If 'no', briefly state why the matrix B cannot exist.

`Yes.`

`Example:` $\underbrace{\begin{bmatrix} 1 & 1 \\ 5 & 5 \\ 9 & 9 \\ 13 & 13 \end{bmatrix}}_{B} x = \underbrace{\begin{bmatrix} 1 \\ 5 \\ 9 \\ 13 \end{bmatrix}}_{u+v}$ `has infinitely many solutions;`

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$, $x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, `etc.`

$$\textbf{3.}\ \text{Let } A = L\underbrace{\begin{bmatrix} 2 & -2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{U}$$ where A is <u>symmetric</u> and L is a unit lower triangular matrix. (Reminder: A unit lower triangular matrix has ones on the diagonal and zeros above the diagonal.)

[6] **3a.** Find a nonzero solution to $Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

`Example:  Find the "special" solution to` $Ux = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\underbrace{\begin{bmatrix} 2 & -2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{U}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ `has the "special" solution,` $x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

[12] **3b.** Find the unit lower triangular matrix L.

`If` A `is symmetric,` $A = L\underbrace{\begin{bmatrix} 2 & -2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{DL^T}$ `where` $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ `and` $L^T = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

`Therefore,` $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$.

[16] **3c.** Give the complete solution to $Ax = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$. Identify $x_{\text{homogeneous}}$.

`If the LU-factorization is available, we can solve` $Ax = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}}_{b}$

`as` $L\underbrace{Ux}_{c} = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}}_{b}$.

$$Lc = b: \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \texttt{ gives } c = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}.$$

$$Ux = c: \quad \begin{bmatrix} 2 & -2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \texttt{ gives } x_{\text{particular}} = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}.$$

$$x_{\text{complete}} = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + \underbrace{x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{x_{\text{homogeneous}}}.$$

[6] **4a.** Given $\underbrace{\begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \\ -3 & -9 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 4 & 3 & 3 \\ -1 & -1 & -1 \\ -3 & 0 & -1 \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$, find the inverse of $\underbrace{\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & -9 \\ 0 & 1 & -1 \end{bmatrix}}_{A^T}$.

$$\left(A^T\right)^{-1} = (A^{-1})^T = \begin{bmatrix} 4 & -1 & -3 \\ 3 & -1 & 0 \\ 3 & -1 & -1 \end{bmatrix}.$$

[10] **4b.** Can you find a matrix A for which $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is a basis for the column space,

and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for the nullspace?

If 'yes', give the matrix A.
If 'no', explain why the matrix A cannot exist.

No.

A is a $m = 3$ by $n = 3$ matrix.
The dimension of the column space (r) plus the dimension of the nullspace (n-r) must equal n.
However, in the problem statement, we find that $1 + 1 \neq 3$.

If the dimension of the column space is 1, the dimension of the nullspace is 2.
If the dimension of the nullspace is 1, the dimension of the column space is 2.