# 18.06 Problem Set 6

Due Wednesday, Oct. 25, 2006 at **4:00 p.m.** in 2-106

**Problem 1** *Wednesday 10/18*

*Some theory of orthogonal matrices:*
(a) Show that, if two matrices $Q_1$ and $Q_2$ are orthogonal, then their product $Q_1 Q_2$ is orthogonal.[1]
(b) Show that, if $Q$ is a *square* orthogonal matrix, then its transpose $Q^\top$ is also orthogonal. *(Hint: $Q$ has an inverse. What is $Q^{-1}$?)*
(c) Is the transpose of a *non-square* orthogonal matrix still orthogonal? Explain why or why not.

***Solution 1***

(a) To see if a matrix $Q$ is orthogonal, we can just check $Q^\top Q \stackrel{?}{=} I\ldots$ for $Q_1Q_2$, we check $(Q_1Q_2)^\top Q_1Q_2 = Q_2^\top Q_1^\top Q_1 Q_2 = Q_2^\top I Q_2 = I$ ✓.
(b) To see if $Q^\top$ is orthogonal, we can just check $(Q^\top)^\top Q^\top = QQ^\top \stackrel{?}{=} I\ldots$ $Q$ is orthogonal, so $Q^\top Q = I$. That means $Q^\top$ is the inverse of $Q$, and so $QQ^\top = I$ also. ✓
(c) If $Q$ is orthogonal, then its columns are linearly independent, so it has full column rank. But then $Q^\top$ has full *row* rank, and can't have full column rank unless it's square. So its columns are linearly dependent.

**Problem 2** *Wednesday 10/18*

(a) Do Gram-Schmidt elimination on A=$\begin{bmatrix} 1 & 5 & 3 \\ 2 & -2 & -1 \\ 3 & -5 & 9 \end{bmatrix}$ to find $A = QR$.

(b) *(You can do this by hand, but I recommend Matlab.)* Find $A^\top$A, and then factor this (symmetric) matrix in your choice of two ways:

- $LDU$-factorization $A^\top A = LDL^\top$ ($U = L^\top$, since $A^\top A$ is symmetric)[2]
- Cholesky factorization $A^\top A = LL^\top$ (a variant of $LDL^\top$; the $L$ is different!)[3]

(c) How are $L^\top$ and $R$ related? *Gram-Schmidt on $A$ is just elimination on $A^\top A$!*

***Solution 2***

(a) By hand, I got $Q = \begin{bmatrix} 1/\sqrt{14} & 6/\sqrt{40} & 1/\sqrt{35} \\ 2/\sqrt{14} & 0 & -5/\sqrt{35} \\ 3/\sqrt{14} & -2/\sqrt{40} & 3/\sqrt{35} \end{bmatrix}$ and $R = \begin{bmatrix} \sqrt{14} & -\sqrt{14} & 2\sqrt{14} \\ 0 & \sqrt{40} & 0 \\ 0 & 0 & \sqrt{35} \end{bmatrix}$.
(b) $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 35 \end{bmatrix}$. For Cholesky:

[1] Remember that an "orthogonal matrix" is really an *orthonormal* matrix; its columns are orthogonal *and* normalized.

[2] The `slu.m` Teaching Code only gives you $A^\top A = LU$; you'll have to calculate $D$ on your own. Here's one way: extract the diagonal of $U$ into a vector $d$ with `d = diag(U)`, then make a diagonal matrix out of $d$ with `D=diag(d)` *(same function name, different functions!)*.

[3] If $D$ has only positive pivots, then we can take its square root and write $LDL^\top$ even more simply, as $(L\sqrt{D})(\sqrt{D}^\top L^\top) = L_1 L_1^\top$, where $L_1 = (L\sqrt{D})$. That's the *Cholesky factorization*, which you can get in Matlab by `L=chol(A'A)`.

```
>> L=chol(S)
L =
    3.7417   -3.7417    7.4833
         0    6.3246    0.0000
         0         0    5.9161
```

(This is $\begin{bmatrix} \sqrt{14} & -\sqrt{14} & 2\sqrt{14} \\ 0 & \sqrt{40} & 0 \\ 0 & 0 & \sqrt{35} \end{bmatrix}$, if you compute the exact values by hand.)

(c) If you used the Cholesky $L$, then $L^\top = R$ exactly! *(Or almost exactly: if you try* `[Q,R]=qr(A)` *in Matlab, the first row of $R$ is negated, because they used $-q_1$ in $Q$ where we used $q_1$.)* If you used $LDL^\top$, then $R = \sqrt{D}L^\top$:

$$\begin{bmatrix} \sqrt{14} & 0 & 0 \\ 0 & \sqrt{40} & 0 \\ 0 & 0 & \sqrt{35} \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{14} & -\sqrt{14} & 2\sqrt{14} \\ 0 & \sqrt{40} & 0 \\ 0 & 0 & \sqrt{35} \end{bmatrix}$$

**Problem 3** *Wednesday 10/18*

(a) Write down the matrix $P$ representing the projection onto the plane *perpendicular* to $a = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$.
(Hint: $P = I - P_1$, where $P_1$ is the projection ____ .)
(b) Now write down the matrix $Q$ representing the *reflection* through that plane. *($Q$ is sometimes called a "Householder matrix".)* $Q = I - 2vv^\top$ for some vector $v =$ ____ .
(c) Show $Q$ is an orthogonal matrix.

***Solution 3***

(a) $P = I - P_1$, where $P_1$ is the projection onto the line along $a = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$.

So $P_1 = \dfrac{aa^\top}{a^\top a} = \frac{1}{9}\begin{bmatrix} 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix}$ and $P = I - P_1 = \frac{1}{9}\begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}$.

(b) $Q = I - 2P_1 = \frac{1}{9}\begin{bmatrix} 7 & -4 & 4 \\ -4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}$.

Since $P_1 = \frac{1}{9}a^\top a = v^\top v$ where $v = a/3 = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$, $Q = I - 2vv^\top$ where $v = a/3 = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$.

(c) You can check the dot products of the columns $q_i$ and make sure $q_i^\top q_i = 1$, $q_i^\top q_j = 0$.
*Here's another way:* since $Q = I - 2vv^\top$, $Q^\top Q = (I^\top - 2(vv^\top)^\top)(I - 2vv^\top) = (I - 2vv^\top)(I - 2vv^\top) = Q^2$, and $Q$ is a reflection so $Q^2 = I$. *(Two reflections bring us back where we started!)*[4]

**Problem 4** *Friday 10/20*

Do Problem #32 from section 5.1 in your book. *(Uses Matlab.)*

***Solution 4***

It took me until $n = 600$ to get Matlab's determinant function to overflow with an `Inf`(inity). I should have asked you for more numbers!

```
>> det(rand(50))
```

[4] We've actually just shown that *any* reflection matrix, not just this $Q$, is orthogonal.

```
ans =
   6.5910e+05

>> det(rand(100))
ans =
   5.1448e+25

>> det(rand(200))
ans =
   2.4949e+80

>> det(rand(400))
ans =
  1.1397e+219

>> det(rand(500))
ans =
  3.4327e+298

>> det(rand(600))
ans =
   Inf
```

If we really wanted to be sure these were "typical", we might run each of these a few more times. But these look good to me.
*(The point of this is that determinants get really big, really fast! It's hard to do computations involving determinants when $n$ gets big.)*

**Problem 5** *Friday 10/20*

Do Problem #24 from section 5.1 in your book.

***Solution 5***

$\det L = 1$, $\det U = \det A = -6$, $\det U^{-1}L^{-1} = -1/6$, and $\det U^{-1}L^{-1}A = \det I = 1$.

**Problem 6** *Monday 10/23, but you can start on Friday*

Do Problem #14 from section 5.1 in your book.
Now compute these determinants using the big formula (with $n!$ terms) or cofactor expansion (your choice). Which is easier?
*(The determinants are* $\det(A) = 36, \det(B) = 5$, *if you want to check your work. Note that* $\det(A)$ *is wrong in the back of the book—sorry!)*

***Solution 6***

(Matrix A, elimination) $\det A = \det \begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{2} & 0 & 1 \\ 0 & 2 & 3 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} = \det \begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{2} & 0 & 1 \\ 0 & 0 & \boxed{3} & 2 \\ 0 & 0 & 0 & \boxed{6} \end{bmatrix} = 1 \cdot 2 \cdot 3 \cdot 6 = 36$

(Matrix B, elimination)

$$\begin{aligned}
\det B &= \det \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \\
&= \det \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \\
&= \det \begin{bmatrix} \boxed{2} & -1 & 0 & 0 \\ 0 & \boxed{3/2} & -1 & 0 \\ 0 & 0 & \boxed{4/3} & -1 \\ 0 & 0 & 0 & \boxed{5/4} \end{bmatrix} \\
&= 2(3/2)(4/3)(5/4) \\
&= 5
\end{aligned}$$

(Matrix A, cofactors)

$$\begin{aligned}
\det A &= \det \begin{bmatrix} 6 & 6 & 1 \\ 0 & 0 & 3 \\ 2 & 0 & 7 \end{bmatrix} - 2\det \begin{bmatrix} 2 & 6 & 1 \\ -1 & 0 & 3 \\ 0 & 0 & 7 \end{bmatrix} + 3\det \begin{bmatrix} 2 & 6 & 1 \\ -1 & 0 & 3 \\ 0 & 2 & 7 \end{bmatrix} - 0 \\
&= (6\det \begin{bmatrix} 0 & 3 \\ 0 & 7 \end{bmatrix} - 6\det \begin{bmatrix} 0 & 3 \\ 2 & 7 \end{bmatrix} + \det \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}) \\
&\quad -2(2\det \begin{bmatrix} 0 & 3 \\ 0 & 7 \end{bmatrix} - 6\det \begin{bmatrix} -1 & 3 \\ 0 & 7 \end{bmatrix} + \det \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}) \\
&\quad +3(2\det \begin{bmatrix} 0 & 3 \\ 2 & 7 \end{bmatrix} - 6\det \begin{bmatrix} -1 & 3 \\ 0 & 7 \end{bmatrix} + \det \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}) \\
&= (6(0) - 6(-6) + (0)) - 2(2(0) - 6(-7) + (0)) + 3(2(-6) - 6(-7) + (-2)) \\
&= (36) - 2(-42) + 3(28) \\
&= 36
\end{aligned}$$

(Matrix B, the big formula)

$$\begin{aligned}
\det B &= \underbrace{(2)(2)(2)(2)}_{(1,2,3,4)} - \underbrace{(2)(2)(-1)(-1)}_{(1,2,4,3)} - \underbrace{(2)(-1)(-1)(2)}_{(1,3,2,4)} \\
&\quad - \underbrace{(-1)(-1)(2)(2)}_{(2,1,3,4)} + \underbrace{(-1)(-1)(-1)(-1)}_{(2,1,4,3)} + \underbrace{(0) \pm \ldots \pm (0)}_{\text{all others}} \\
&= 16 - 4 - 4 - 4 + 1 \\
&= 5
\end{aligned}$$

I thought elimination was easier for $\det(A)$. For $\det(B)$, they both had strengths: the big formula had lots of terms to check, but elimination had fractions. What did you think?

**Problem 7** *Monday 10/23, but you can start on Friday*

Suppose we fit the quadratic $y = C + Dt + Et^2$ to three points $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ by least-squares.

(a) Write down the least-squares matrix $V$. $V \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ *($V$ is called the "Vandermonde matrix".)*

(b) Find $\det V$ by row operations.

(c) Now write down the big formula (with 3! terms) for $\det V$.
(d) Here's a trick for finding $\det V$ easily: we see from the big formula that $\det V$ is a polynomial in $a_1, a_2, a_3$, and all 3! terms have degree ____. Now find the factors of $\det V$. The first two rows are equal when ____, so when ____, $\det V = 0$. Name a factor of $\det V$: ____. Now name two more factors of $\det V$, for the other two pairs of rows: ____, ____. How do you know any remaining factor of $\det(V)$ is constant? Now find the constant, and you're done!
(e) When can we fit a quadratic *exactly* through three points?

***Solution 7***

(a) $V = \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{bmatrix}$
(b) $V \rightsquigarrow \begin{bmatrix} 1 & a_1 & a_1^2 \\ 0 & a_2 - a_1 & a_2^2 - a_1^2 \\ 0 & a_3 - a_1 & a_3^2 - a_1^2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & a_1 & a_1^2 \\ 0 & a_2 - a_1 & a_2^2 - a_1^2 \\ 0 & 0 & a_3^2 - a_3a_2 - a_3a_1 + a_2a_1 \end{bmatrix}$ so $\det V = (a_2 - a_1)(a_3^2 - a_3a_2 - a_3a_1 + a_2a_1)$. That last bit factors, as we'll see in (d).
(c) $\det V = a_1a_2^2 - a_1a_3^2 - a_2a_1^2 + a_2a_3^2 + a_3a_1^2 - a_3a_2^2$.
(d) All the terms have degree 3. When $a_1 = a_2$, the first two rows are equal so $\det(V) = 0$; that means $(a_1 - a_2)$ is a factor of $\det V$. The other two factors are $(a_2 - a_3)$ and $(a_1 - a_3)$, so $\det V = (\text{const.})(a_1 - a_2)(a_2 - a_3)(a_1 - a_3)$ — all our factors together have degree 3, so the remaining factor must be a constant! Check that constant: put $a_1 = 0$, $a_2 = 1$, $a_3 = 2$ and get $\det = +2$ so the constant is $-1$.
(e) The equation is solvable exactly whenever $\det(V) \neq 0$, which happens when the three $t$-values $a_1, a_2, a_3$ are distinct.

**Problem 8** *Monday 10/23*

Do Problem #25 from section 5.2 in your book.

***Solution 8***

(a) *Think about the big formula.* If we don't pick any 0 entries, then the first two columns are picked from $A$ and the last two rows are from $D$. We can't pick any columns or rows from $B$, because there aren't any left. *Or think about elimination.* Use the pivots in $D$ to eliminate the entries in $B$. (If $D$ is missing pivots, then we have a zero row, so the det is zero!)
(b) Try $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, which has $\det = -1$ but all the blocks have $\det = 0$!
(c) The matrix above works here too, since $AD = CB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

**Problem 9** *Monday 10/23*

Do Problem #14 from section 5.2 in your book.

***Solution 9***

$\det(B_4) = +1 \det \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & & \\ & -1 & \boxed{\text{-1}} & \end{bmatrix} + 2 \det \begin{bmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} = +1 \det \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} + 2 \det \begin{bmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} = -\det(B_2) + 2\det(B_3)$. *(Put in extra rows, and we have* $\det(B_{n+2}) = -\det(B_n) + 2\det(B_{n+1})$ *for any* $n > 0$.*)* Since $\det(B_3) = \det(B_2) = 1$, $B_4 = 1$ also.