18.06 Problem Set 4

Due Wednesday, Oct. 11, 2006 at **4:00 p.m.** in 2-106

Problem 1 *Monday 10/2*

Consider the eight vectors $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) List all of the one-element, linearly dependent sets[1].

(b) What are the two-element, linearly dependent sets?

(c) Find a three-element set spanning a subspace of dimension three. Can you find a three-element set spanning a subspace of dimension two? One? Zero?

(d) Which four-element sets are linearly dependent? Explain why.

Problem 2 *Monday 10/2*

Consider the matrix $A = \begin{bmatrix} 1 & 0 & a \\ 2 & -1 & b \\ 1 & 1 & c \\ -2 & 1 & d \end{bmatrix}$.

(a) Which vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ will make the columns of A linearly dependent?

(b) Which vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ will make the columns of A a basis for $\left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : y + w = 0 \right\}$?

(c) For $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \\ 2 \end{bmatrix}$, compute a basis for the four subspaces.

Problem 3 *Monday 10/2*

Do Problem #5 from section 3.5 in your book.

Problem 4 *Monday 10/2*

Do Problem #24 from section 3.5 in your book.

Problem 5 *Monday 10/2*

Do Problem #37 from section 3.5 in your book.

Problem 6 *Monday 10/2*

Do Problem #19 from section 3.6 in your book.

[1] That is, a collection, list, or sequence of vectors ("set": $S = \{u, v, \ldots\}$) containing exactly one vector ("one-element": so $S = \{u\}$) and that sequence is not linearly independent. *(the matrix A having columns $u, \ldots,$ is not $\ldots$?)* Technically, a set of vectors is different from a sequence of vectors in that (a) order doesn't matter, and (b) duplicates don't count (for example, $\{u, v\} = \{v, u\} = \{u, v, u, u\}$ are all the same set, with two elements u, v) — don't worry about the technical definition if you don't already know it, though. If you want, you may pretend "set" here means "sequence" or "list" of vectors.

Problem 7 *Monday 10/2*

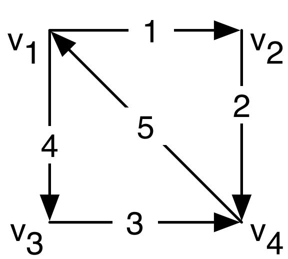
Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) What is the rank of A? What are the dimensions of the four subspaces?
(b) Give a basis for each of the four subspaces.
(c) Now, for each of the four subspaces, find the set of equations that all vectors in the subspace must satisfy. (For example, if $Ax = b$ for some x, what are the conditions on the components b_i of b?)
(d) Give the complete solution to $A^{\mathsf{T}} y = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}$.

Problem 8 *Monday 10/2*

Using Matlab, take some random 3-by-3 matrices (try using the `rand(m,n)` function) and look at their four subspaces. (A convenient way to calculate the subspaces is the `fourbase.m` teaching code; type in `type fourbase` at the Matlab prompt for information on how to use it.[2]) What are the dimensions of the four subspaces for a "typical" 3-by-3 matrix? Can you explain why? *(Hint: what are the odds a pivot is exactly zero?)*
Now try 3-by-5 matrices. What are the dimensions of the four subspaces now? Now guess what dimensions the four subspaces of a random m-by-n matrix will most likely have.

Problem 9 *Friday 10/6*



(a) Find an ~~adjacency~~ incidence[3] matrix A for the graph above.
(b) Find one solution to $Ax = 0$ and two linearly independent solutions to $A^{\mathsf{T}} y = 0$.
(c) What conditions on the components of b do we need for $Ax = b$ to have a solution? Tell which of Kirchhoff's laws this illustrates. What are the "currents"? What are the "voltages"?
(d) Compute $A^{\mathsf{T}} A$. You get positive numbers on the diagonal — these numbers count the number of ____ each node has. When are the off-diagonal entries -1, and when are they zero?
(e) What is $N(A^{\mathsf{T}} A)$? Why does $A^{\mathsf{T}} Ax = f$ have a solution only when $f_1 + f_2 + f_3 + f_4 = 0$?

[2]If you need to download the file `fourbase.m` from the Web site, don't forget to put it in the current directory where Matlab can find it.
[3]Thanks to J. Tang for the correction!