18.06 Problem Set 4

Due Wednesday, Oct. 11, 2006 at **4:00 p.m.** in 2-106

Problem 1 *Monday 10/2*

Consider the eight vectors $\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \ldots, \begin{bmatrix}1\\1\\1\end{bmatrix}$.

(a) List all of the one-element, linearly dependent sets formed from these.
(b) What are the two-element, linearly dependent sets?
(c) Find a three-element set spanning a subspace of dimension three. Can you find a three-element set spanning a subspace of dimension two? One? Zero?
(d) Which four-element sets are linearly dependent? Explain why.

Solution 1

(a) Only the zero vector, $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$, is linearly dependent by itself.

(b) Only the sets containing $\begin{bmatrix}0\\0\\0\end{bmatrix}$ and some other vector.

(c) How about $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$? We can replace one of these by $\begin{bmatrix}0\\0\\0\end{bmatrix}$ to get dimension 2. There are only two vectors on each line (= 1-D subspace), so we can't have a three-element set spanning only a 1-dimensional subspace unless we allow duplicates; if we do, then something like $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$ works. For zero dimensions (a point!), the only way is three copies of the zero vector $\begin{bmatrix}0\\0\\0\end{bmatrix}$.

(d) All four-element sets are linearly dependent in a three-dimensional space.

Problem 2 *Monday 10/2*

Consider the matrix $A = \begin{bmatrix}1 & 0 & a\\2 & -1 & b\\1 & 1 & c\\-2 & 1 & d\end{bmatrix}$.

(a) Which vectors $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ will make the columns of A linearly dependent?

(b) Which vectors $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ will make the columns of A a basis for $\left\{\begin{bmatrix}x\\y\\z\\w\end{bmatrix} : y + w = 0\right\}$?

(c) For $\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}1\\-2\\5\\2\end{bmatrix}$, compute a basis for the four subspaces.

Solution 2

(a) All linear combinations of $\begin{bmatrix}1\\2\\1\\-2\end{bmatrix}$ and $\begin{bmatrix}0\\-1\\1\\1\end{bmatrix}$.

(b) All vectors which are not linear combinations of these, and which satisfy $b+d = 0$. For example,

$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. For the set of all possible examples, $\left\{ A \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + B \begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \end{bmatrix} + C \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} : A \neq 0 \right\}$.

(c) Since $\begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \\ 2 \end{bmatrix}$, we know the first two columns span the column space$\mathbf{C}(A)$ and $\begin{bmatrix} 1 & 4 & -1 \end{bmatrix}^\mathsf{T}$ is in the nullspace $\mathbf{N}(A)$. Since the first two columns are linearly independent, we conclude the first two columns are a basis for $\mathbf{C}(A)$, so A has rank 2 and $\begin{bmatrix} 1 & 4 & -1 \end{bmatrix}^\mathsf{T}$ is a basis for the one-dimensional $\mathbf{N}(A)$.

You can do the same for A^T: the fourth row is the negative of the first, so $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ is in $\mathbf{N}(A^\mathsf{T})$, and the second and third row sum to three times the first, so $\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ is in $\mathbf{N}(A^\mathsf{T})$. These are linearly independent, so the left nullspace$\mathbf{N}(A^\mathsf{T})$ has at least two dimensions. *But the rank of A is 2* — so that's everybody! Those two linearly independent elements form a basis for the left nullspace. *Now do the row space* $\mathbf{C}(A^\mathsf{T})$*:* Any two basis elements will do, say the first and second rows, or the first and third rows (but not the second and fourth rows, because they're linearly dependent!) So one basis would be $\left\{ \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^\mathsf{T}, \begin{bmatrix} 2 & -1 & -2 \end{bmatrix}^\mathsf{T} \right\}$.
Of course, you can also do elimination.

Problem 3 *Monday 10/2*

Do Problem #5 from section 3.5 in your book.

Solution 3

(a) These are linearly independent: the only linear combination $x_1(1,3,2)+x_2(2,1,3)+x_3(3,2,1) = (0,0,0)$ is $x_1 = x_2 = x_3 = 0$. (You can check this by writing this linear combination as the system of equations $Ax = 0$ and then solving by elimination to get $x = 0$.)
(b) These are linearly dependent: $1(1,-3,2) + 1(2,1,-3) + 1(-3,2,1) = (0,0,0)$.

Problem 4 *Monday 10/2*

Do Problem #24 from section 3.5 in your book.

Solution 4

(a) The pivots are in the first two columns, so one possible basis for $\mathbf{C}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}$ and for $\mathbf{C}(U)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$. (Notice they're different!)

(b) Both A and U have the same nullspace $\mathbf{N}(A) = \mathbf{N}(U)$, with basis $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(c) Both A and U have the same row space $\mathbf{C}(A^\mathsf{T}) = \mathbf{C}(U^\mathsf{T})$, with basis $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Problem 5 *Monday 10/2*

Do Problem #37 from section 3.5 in your book.

Solution 5

I used the basis $\{1, x, x^2, x^3\}$ for the first part, and $\{x-1, x^2-1, x^3-1\}$ for the second part.

Problem 6 *Monday 10/2*

Do Problem #19 from section 3.6 in your book.

Solution 6

(a) Elimination gives $\begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2 - 3b_1 \\ 0 & -2 & b_3 - 4b_1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2 - 3b_1 \\ 0 & 0 & \mathbf{b_3 - b_2 - b_1} \end{bmatrix}$, so $\mathbf{N}(A^\mathsf{T})$ has basis $\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$.

(b) Elimination gives $\begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 - 2b_1 \\ 0 & 1 & b_4 - 2b_1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & \mathbf{b_3 - 2b_1} \\ 0 & 0 & \mathbf{b_4 + b_2 - 4b_1} \end{bmatrix}$, so $\mathbf{N}(A^\mathsf{T})$ has basis $\left\{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Problem 7 *Monday 10/2*

Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) What is the rank of A? What are the dimensions of the four subspaces?

(b) Give a basis for each of the four subspaces.

(c) Now, for each of the four subspaces, find the set of equations that all vectors in the subspace must satisfy. (For example, if $Ax = b$ for some x, what are the conditions on the components b_i of b?)

(d) Give the complete solution to $A^\mathsf{T} y = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}$.

Solution 7

(a) A has rank 2, since the left factor (call it L) is invertible, and the right factor (R) has rank 2. So the subspaces have dimensions: $r = 2$ for the row space and column space; $n - r = 2$ for the nullspace, and $m - r = 1$ for the left nullspace.

(b) It's easy to give bases for the four subspaces of R. Look at what that says about the four subspaces of A:

The column space: A's columns are just L times the columns of R. Since a basis for $\mathbf{C}(R)$ is $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$, the corresponding basis for $\mathbf{C}(A)$ must be $\left\{L\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, L\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}\right\}$.

The nullspace: $Ax = 0$ if and only if $Rx = 0$, so $\mathbf{N}(A) = \mathbf{N}(R) = \left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 0 \\ 1 \end{bmatrix}\right\}$.

The row space: Since $\mathbf{N}(A) = \mathbf{N}(R)$, the row spaces (their orthogonal complements!) must also be equal: $\mathbf{C}(A^\mathsf{T}) = \mathbf{C}(R^\mathsf{T}) = \left\{\begin{bmatrix} 1 & 2 & 0 & 3 \end{bmatrix}^\mathsf{T}, \begin{bmatrix} 0 & 0 & 1 & 4 \end{bmatrix}^\mathsf{T}\right\}$.

The left nullspace: You can do this by the method of Worked Example 3.6A, the method you used above in Problem #19, or by orthogonality to the column space. The basis for $\mathbf{N}(R^\mathsf{T})$ is $\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$, so the basis for $\mathbf{N}(A^\mathsf{T})$ is $\left\{(L^{-1})^\mathsf{T}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\} = \left\{\begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}\right\}$.

(c) Here are equations defining each of the four subspaces. (Note that there are other possible sets of equations, which can be got by taking linear combinations of the given equations.)
The column space: $-10b_1 + 3b_2 + b_3 = 0$, by elimination on $[\,A \quad b\,]$.
The row space: $-2c_1 + c_2 = 0, -3c_1 - 4c_2 + c_4 = 0$; I used orthogonality to the nullspace, but you could use elimination on $\left[\,A^{\mathsf{T}} \quad c\,\right]$.
The nullspace: The equations defining $Ax = 0$ are just the rows of A: $x_1+x_2+3x_4 = 0, x_3+4x_4 = 0$.
The left nullspace: equations are the columns (pivot columns suffice): $y_1+2y_2+4y_3 = 0, y_2-3y_3 = 0$.
(d) Once we have a particular solution (try $y_p = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$), the complete solution is just $y = y_p + \mathbf{N}(A^{\mathsf{T}})$. (I got this particular solution by back-substitution: $A^{\mathsf{T}}y = b$ becomes $R^{\mathsf{T}}L^{\mathsf{T}}y = b$. So let $L^{\mathsf{T}}y = z$ and solve $R^{\mathsf{T}}z = b$ (solution: $z = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$), and then find y.)

Problem 8 *Monday 10/2*

Using Matlab, take some random 3-by-3 matrices (try using the `rand(m,n)` function) and look at their four subspaces. (A convenient way to calculate the subspaces is the `fourbase.m` teaching code; type in `type fourbase` at the Matlab prompt for information on how to use it.[1]) What are the dimensions of the four subspaces for a "typical" 3-by-3 matrix? Can you explain why? *(Hint: what are the odds a pivot is exactly zero?)*
Now try 3-by-5 matrices. What are the dimensions of the four subspaces now? Now guess what dimensions the four subspaces of a random m-by-n matrix will most likely have.

Solution 8

Here's one matrix:

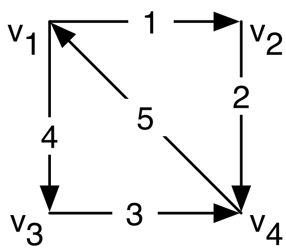
```
>> [R,N,C,L] = fourbase(rand(3,5))

R =
    1.0000         0         0
         0    1.0000         0
         0         0    1.0000
    1.2136    1.2683   -0.1678
   -2.3187    1.9189    0.6334


N =
   -1.2136    2.3187
   -1.2683   -1.9189
    0.1678   -0.6334
    1.0000         0
         0    1.0000


C =
    0.1389    0.6038    0.0153
    0.2028    0.2722    0.7468
    0.1987    0.1988    0.4451
```

[1] If you need to download the file `fourbase.m` from the Web site, don't forget to put it in the current directory where Matlab can find it.

```
L =
   Empty matrix: 3-by-0
```

Unless we're very unlucky, we always get similar results: the row and column space have dimension 3, the nullspace has dimension 2, and the left nullspace has dimension 0 (containing only the zero vector). *(In other words, almost all random matrices have full rank.)* The chance that the number I pick for any diagonal entry `A(i:i)` is exactly zero after elimination is negligible, so we can choose the diagonal entries as our pivots.[2]

Problem 9 *Friday 10/6*



(a) Find an incidence matrix A for the graph above.
(b) Find one solution to $Ax = 0$ and two linearly independent solutions to $A^\mathsf{T} y = 0$.
(c) What conditions on the components of b do we need for $Ax = b$ to have a solution? Tell which of Kirchhoff's laws this illustrates. What are the "currents"? What are the "voltages"?
(d) Compute $A^\mathsf{T} A$. You get positive numbers on the diagonal — these numbers count the number of ____ each node has. When are the off-diagonal entries -1, and when are they zero?
(e) What is $N(A^\mathsf{T} A)$? Why does $A^\mathsf{T} Ax = f$ have a solution only when $f_1 + f_2 + f_3 + f_4 = 0$?

Solution 9

(a) $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$

(b) $A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0$ and $A^\mathsf{T} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = A^\mathsf{T} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0$, for example. (Since A has rank 3, these are a basis!)

(c) b is in the column space $\mathbf{C}(A)$, which is orthogonal to the left nullspace $\mathbf{N}(A^\mathsf{T})$. Just check basis vectors: $b^\mathsf{T} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0$, $b^\mathsf{T} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0$, or in other words $b_1 + b_2 + b_5 = 0$, $b_3 + b_4 + b_5 = 0$.
This is *Kirchhoff's voltage law:* the voltage change b_i on each edge i sums to zero around each loop. Here, the components of x are the voltages at each node, and the components of b are the voltage

[2] Suppose you've already picked the earlier entries of `A`, and then you pick `A(i:i)` at random — let's say there's less than a $\frac{1}{1000}$ chance `A(i:i)` will be any particular c. (In symbols, $\Pr(\texttt{A(i:i)} = c) \leq \frac{1}{1000}$.) *Now do elimination:* this subtracts some multiple of each of the previous rows from row i, to make `A(i:1)` through `A(i:i-1)` zero. *You haven't even looked at entry `A(i:i)` yet,* so whatever c you subtract from `A(i:i)` only depends on the earlier entries, *not on `A(i:i)`*. So `A(i:i)` isn't a pivot *if and only if* `A(i:i)` equals that particular c. And that means $\Pr(\texttt{A(i:i)}\text{isn't a pivot}) = \Pr(\texttt{A(i:i)} = c) \leq \frac{1}{1000}$.

differences along each edge. Voltage difference is proportional to current flow *(remember $\Delta V = iR$ from 8.02?)* so the components of b represent currents through each edge.

(d) $A^{\mathsf{T}}A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$. The diagonal entries count the number of *edges* each node has. The off-diagonal entries s_{ij} are -1 if nodes i and j are connected by an edge, and 0 if not.

(e) $\mathbf{N}(A^{\mathsf{T}}A) = \mathbf{N}(A)$ is again the set of multiples of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. *Since $A^{\mathsf{T}}A$ is symmetric, the left nullspace is the same!* When $A^{\mathsf{T}}A = f$ has a solution, f is in the column space $\mathbf{C}(A^{\mathsf{T}}A)$, which is orthogonal to the left nullspace: so $f^{\mathsf{T}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$=0.