18.06 Problem Set 1

Due Wednesday, Sept. 13, 2006 at **4:00 p.m.** in 2-106

Problem 1 *Wednesday 9/06*

Go read the Worked Examples 2.1A and 2.1B (page 29).
(You don't have to hand anything in for this problem.)

Problem 2 *Wednesday 9/06*

Write the product $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \pi \\ e \end{bmatrix}$ in two ways:
(a) as dot products of the rows with the column vector
(b) as a linear combination of the columns.

Solution 2

(a) $\begin{bmatrix} [1 \quad 2] \cdot \begin{bmatrix} \pi \\ e \end{bmatrix} \\ [3 \quad 4] \cdot \begin{bmatrix} \pi \\ e \end{bmatrix} \end{bmatrix}$ or $\begin{bmatrix} 1 \cdot \pi + 2 \cdot e \\ 3 \cdot \pi + 4 \cdot e \end{bmatrix}$

(b) $\pi \begin{bmatrix} 1 \\ 3 \end{bmatrix} + e \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Problem 3 *Wednesday 9/06*

(a) What matrix A takes $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$?

(b) What is $A \begin{bmatrix} 1 \\ 2 \end{bmatrix}$?

Solution 3

(a) $A = \begin{bmatrix} 3 & 1 \\ 2 & 7 \end{bmatrix}$

(b) $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 16 \end{bmatrix}$.

Problem 4 *Wednesday 9/06*

Do Problem #25 from section 2.1 in your book.

Solution 4

`A*v` gives $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$, and `v'*v` gives $9 + 16 + 25 = 50$. `v*A` is an error, of course: try it!

```
>> v*A
??? Error using ==> mtimes
Inner matrix dimensions must agree.
```

Problem 5 *Wednesday 9/06*

Let's practice using Matlab by multiplying a random pair of upper-triangular matrices. (Hint: you can type `diary` at the beginning of your session to save a transcript.)

We'll need two matrices. *First you pick one:* Let `A=[a b c; 0 d e; 0 0 f]`, where `a...f` are six of your favorite *nonzero* numbers. *Now let the computer pick one:* `B=rand(3,3)` gives us a random 3-by-3 matrix; we can zero out the extra coefficients one-by-one by typing e.g. `B(3,2)=0`, or all at once by keeping only the upper-triangular part `B=triu(B)`
Now compute `A*B` *and* `B*A`. What shape is this new matrix? Are AB and BA equal?

Solution 5

```
>> A=[-1 2 -3;0 -4 5;0 0 6]
A =
    -1     2    -3
     0    -4     5
     0     0     6

>> B=rand(3,3)
B =
    0.9501    0.4860    0.4565
    0.2311    0.8913    0.0185
    0.6068    0.7621    0.8214

>> B(2,1)=0;B(3,1)=0;B(3,2)=0
B =
    0.9501    0.4860    0.4565
         0    0.8913    0.0185
         0         0    0.8214

>> A*B
ans =
   -0.9501    1.2966   -2.8837
         0   -3.5652    4.0330
         0         0    4.9284
```

The product of these two upper-triangular matrices is upper-triangular.
(We can prove this in general; see exercise 27 in section 2.4.)

Problem 6 *Friday 9/08*

(a) Write examples of systems $A\vec{x} = \vec{b}$ where A is a 3-by-3 matrix and:

1. in the row picture, all three planes are parallel but distinct
2. all three planes are equal
3. the three planes meet in a common line
4. in the column picture, $\vec{b}$ is a linear combination of the first two columns of A.
5. $\vec{b}$ is not a linear combination of the columns of A.

(b) How many solutions for each of these? Describe the shape (point, line, ...) of each solution set.
(c) Reduce each by elimination (you need not back-substitute) and check your answer.
Circle the pivots.

Solution 6

(a) Answers may vary, but here are some examples:

1. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ -1 & -1 & 5 \end{bmatrix} \vec{x} = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 1 & -1 & -3 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$

(b) Counting solutions:

1. There is no solution, since the planes never meet.
2. Any of the (∞ly many) points on that plane (here, $x + 2y + 3z = 5$) is a solution.
3. Here, the solution set is the common line (infinitely many points).
4. The number of solutions depends on *how many* ways we can form $\vec{b}$ as a linear combination $A\vec{x}$ of columns of A. (So there's at least one solution, but there could be more if your matrix A was singular.)
5. There is no solution, since no linear combination of columns $A\vec{x}$ ever yields $\vec{b}$.

(c) Elimination:

1. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 5 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 \end{array}\right]$ which clearly has no solution.

2. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 5 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so any $\vec{x}$ with $x + 2y + 3z = 5$ is a solution.

3. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & -1 & -1 & 1 \\ 1 & 0 & 1 & 7 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$; the line defined by the first two planes $x + 2y + 3z = 5$ and $-y - z = 1$ is the line $\vec{x} = \vec{a} + t\vec{r}$, where $\vec{a} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{r} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

4. For this example, $\left[\begin{array}{ccc|c} 1 & 2 & 3 & -3 \\ 0 & 1 & 4 & -2 \\ -1 & -1 & 5 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & -3 \\ 0 & 1 & 4 & -2 \\ 0 & 1 & 8 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & -3 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 4 & 0 \end{array}\right]$; this example is nonsingular, with the unique solution $\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$.

5. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 \\ 1 & -1 & -3 & 5 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 6 \end{array}\right]$ has no solution.

Problem 7 *Friday 9/08*

Do Problem #6 from section 2.2 in your book.

Solution 7

You need to choose $b = 4$ and $g = 32$. Anything of the form $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} + c \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is a solution.

Problem 8 *Friday 9/08*

Consider the system of equations

$$\begin{aligned} 2x + y + z &= -1 \\ x - z &= 0 \\ 6x + 2y + z &= -1 \end{aligned}$$

Solve this system. (Eliminate, then back-substitute.)
Circle the pivots as you find them.
Write down the elimination matrices E_{21}, E_{31}, E_{32} you used.

Solution 8

Step 0: Our initial system (written as an augmented matrix $\begin{bmatrix} A & \vec{b} \end{bmatrix}$).

$$\begin{bmatrix} A & \vec{b} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 6 & 2 & 1 & -1 \end{bmatrix}$$

Step 1: Eliminate $A(2,1)$ with $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$E_{21} \begin{bmatrix} A & \vec{b} \end{bmatrix} = \begin{bmatrix} \boxed{2} & 1 & 1 & -1 \\ 0 & -1/2 & -3/2 & 1/2 \\ 6 & 2 & 1 & -1 \end{bmatrix}$$

Step 2: Eliminate $A(3,1)$ with $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$.

$$E_{31}E_{21} \begin{bmatrix} A & \vec{b} \end{bmatrix} = \begin{bmatrix} \boxed{2} & 1 & 1 & -1 \\ 0 & \boxed{\text{-1/2}} & -3/2 & 1/2 \\ 0 & -1 & -2 & -2 \end{bmatrix}$$

Step 3: Eliminate $A(3,2)$ with $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$.

$$[U \quad \vec{c}] = E_{32}E_{31}E_{21} \begin{bmatrix} A & \vec{b} \end{bmatrix} = \begin{bmatrix} \boxed{2} & 1 & 1 & -1 \\ 0 & \boxed{\text{-1/2}} & -3/2 & 1/2 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

Step 4: Now back-substitute:

- $z = 1$.
- $-y/2 - 3z/2 = 1/2$ so $y = -4$.
- $2x + y + z = -1$ so $x = 1$.

Problem 9 *Friday 9/08*

Do Problem #22 from section 2.2 in your book.

Solution 9

You can do the calculations by hand, but let's try letting Matlab keep track of our matrix. Let's do Problem 21 first, for comparison:

```
>> A=[2,1,0,0,0;1,2,1,0,0;0,1,2,1,0;0,0,1,2,5]
A =
     2     1     0     0     0
     1     2     1     0     0
     0     1     2     1     0
     0     0     1     2     5
>> A(2,:)=A(2,:)-(1/2)*A(1,:)
A =
    2.0000    1.0000         0         0         0
         0    1.5000    1.0000         0         0
         0    1.0000    2.0000    1.0000         0
         0         0    1.0000    2.0000    5.0000
>> A(3,:)=A(3,:)-(2/3)*A(2,:)
A =
    2.0000    1.0000         0         0         0
         0    1.5000    1.0000         0         0
         0         0    1.3333    1.0000         0
         0         0    1.0000    2.0000    5.0000
>> A(4,:)=A(4,:)-(3/4)*A(3,:)
A =
    2.0000    1.0000         0         0         0
         0    1.5000    1.0000         0         0
         0         0    1.3333    1.0000         0
         0         0         0    1.2500    5.0000
```

So the pivots are $2(=2/1), 3/2, 4/3, 5/4$ (notice the pattern?), and by back-substituting we see $t = 4, z = 3, y = 2, x = 1$.

```
>> A=[2,-1,0,0,0;-1,2,-1,0,0;0,-1,2,-1,0;0,0,-1,2,5]
A =
     2     1     0     0     0
     1     2     1     0     0
     0     1     2     1     0
     0     0     1     2     5
>> A(2,:)=A(2,:)+(1/2)*A(1,:)
A =
    2.0000   -1.0000         0         0         0
         0    1.5000   -1.0000         0         0
         0   -1.0000    2.0000   -1.0000         0
         0         0   -1.0000    2.0000    5.0000
>> A(3,:)=A(3,:)+(2/3)*A(2,:)
A =
    2.0000   -1.0000         0         0         0
         0    1.5000   -1.0000         0         0
         0         0    1.3333   -1.0000         0
         0         0   -1.0000    2.0000    5.0000
>> A(4,:)=A(4,:)+(3/4)*A(3,:)
```

```
A =
    2.0000   -1.0000         0         0         0
         0    1.5000   -1.0000         0         0
         0         0    1.3333   -1.0000         0
         0         0         0    1.2500    5.0000
```

The pivots are the same, but now we get alternating signs: $t = 4, z = -3, y = 2, x = -1$.

Problem 10 *Monday 9/11*

Consider the matrices $A = \begin{bmatrix} 0 & 1 & 4 \\ -2 & 3 & 6 \\ 2 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 0 & 6 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & -1 \\ 4 & 2 \\ 0 & 1 \end{bmatrix}$.

(a) Find AB and AC.
(b) What happens?
(c) Why does this tell you A is not invertible?

Solution 10

(a) $AB = \begin{bmatrix} 4 & 6 \\ 12 & 14 \\ -4 & -2 \end{bmatrix}$ . . . and so is AC.
(b) They're equal!
(c) If A had an inverse, then $A^{-1}AB = A^{-1}AC$ would mean $B = C$.

Problem 11 *Monday 9/11*

Do Problem #35 from section 2.4 in your book.

Solution 11

The product is $\begin{bmatrix} A & B \\ 0 & S \end{bmatrix}$, where $S = D - CA^{-1}B$.