18.06 Professor Strang Quiz 3 December 5, 2005

Grading

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2

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Your PRINTED name is: ____________________

Please circle your recitation:

1)	M 2	2-131	P. Lee	2-087	2-1193	`lee`
2)	M 2	2-132	T. Lawson	4-182	8-6895	`tlawson`
4)	T 10	2-132	P-O. Persson	2-363A	3-4989	`persson`
5)	T 11	2-131	P-O. Persson	2-363A	3-4989	`persson`
6)	T 11	2-132	P. Pylyavskyy	2-333	3-7826	`pasha`
7)	T 12	2-132	T. Lawson	4-182	8-6895	`tlawson`
8)	T 12	2-131	P. Pylyavskyy	2-333	3-7826	`pasha`
9)	T 1	2-132	A. Chan	2-588	3-4110	`alicec`
10)	T 1	2-131	D. Chebikin	2-333	3-7826	`chebikin`
11)	T 2	2-132	A. Chan	2-588	3-4110	`alicec`
12)	T 3	2-132	T. Lawson	4-182	8-6895	`tlawson`

1 (30 pts.) a) Find the eigenvalues and eigenvectors of the Markov matrix

$$A = \begin{bmatrix} .9 & .4 \\ .1 & .6 \end{bmatrix}$$

Solution: Any Markov matrix has eigenvalue $\lambda_1 = 1$; since the trace of A is 1.5, and the eigenvalues of a matrix add up to its trace, the second eigenvalue is $\lambda_2 = .5$. To find the corresponding eigenvectors v_1 and v_2, we look at $A - \lambda_1 I$ and $A - \lambda_2 I$:

$$(A - \lambda_1 I)v_1 = (A - I)v_1 = \begin{bmatrix} -.1 & .4 \\ .1 & -.4 \end{bmatrix} v_1 = 0 \quad \Rightarrow \quad v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix};$$

$$(A - \lambda_2 I)v_2 = (A - .5I)v_2 = \begin{bmatrix} .4 & .4 \\ .1 & .1 \end{bmatrix} v_2 = 0 \quad \Rightarrow \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix};$$

b) What is the limiting value of $A^k \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as the power k goes to infinity?

Solution: We have

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = v_1 + v_2,$$

so

$$A^k \begin{bmatrix} 3 \\ 2 \end{bmatrix} = A^k v_1 + A^k v_2 = v_1 + (.5)^k v_2.$$

Since $(.5)^k$ goes to 0 as k goes to infinity, the limiting value of $A^k \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is $v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Another argument: the steady state eigenvector of A is $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, so the limit of A^k as k goes to infinity is the Markov matrix whose both columns are multiples of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, i.e.

$$A^{\infty} = \begin{bmatrix} .8 & .2 \\ .8 & .2 \end{bmatrix},$$

and the limiting value of $A^k \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is

$$A^{\infty} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

c) What does it mean to say that "*A is similar to B*"?

Is that 2 by 2 matrix A similar (yes or no) to its transpose B?

$$B = \begin{bmatrix} .9 & .1 \\ .4 & .6 \end{bmatrix}$$

Give a reason for your answer.

Solution: Matrices A and B are similar if there exists an invertible matrix M such that $A = M^{-1}BM$. Equivalently, A and B are similar if their Jordan form is the same.

The matrix A^T has the same eigenvalues $\lambda_1 = 1$ and $\lambda_2 = .5$ as A, so both are similar to the same Jordan matrix

$$J = \begin{bmatrix} 1 & \\ & .5 \end{bmatrix}.$$

Thus A is similar to A^T.

2 (40 pts.) This 4 by 4 matrix H is a Hadamard matrix:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Key Properties

$H^{\mathrm{T}} = H$ and $H^2 = 4I$

a) Figure out the eigenvalues of H. Explain your reasoning.

Solution: Suppose $Hv = \lambda v$ for some non-zero vector v. Then $H^2 v = \lambda^2 v = (4I)v = 4v$, so $\lambda^2 = 4$, and thus every eigenvalue of H is equal to either 2 or -2. The trace of H is 0, hence the sum of the eigenvalues of H is 0. We conclude that H has eigenvalues $\lambda = 2, 2, -2, -2$.

b) Figure out H^{-1} and the determinant of H. Explain your reasoning.

Solution: From $H^2 = 4I$ we obtain

$$H^{-1} = \frac{1}{4} H.$$

The determinant of a matrix is the product of its eigenvalues:

$$\det H = 2 \cdot 2 \cdot (-2) \cdot (-2) = 16.$$

c) This matrix S contains three eigenvectors of H. Find a 4th eigenvector x_4 and explain your reasoning:

$$S = \begin{bmatrix} 1 & 1 & 0 & \\ 1 & 0 & -1 & \\ 1 & 0 & 1 & \\ -1 & 1 & 0 & \end{bmatrix}$$

Solution: The first two eigenvectors correspond to $\lambda = 2$, so the missing eigenvector corresponds to $\lambda = -2$. Denote the unknown eigenvector

v_4 by $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. Then

$$(H+2I)v_4 = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3a+b+c+d \\ a+b+c-d \\ a+b+c-d \\ a-b-c+3d \end{bmatrix} = 0$$

The third component of $(H+2I)v_4$ is equal to the second, and the fourth is the sum of the first two, hence we can choose v_4 to be any vector satisfying $3a+b+c+d=0$ and $a+b+c-d=0$ which is not a multiple of the third eigenvector $(0,-1,1,0)$. For example, we can choose

$$v_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}.$$

Note that since H is symmetric and the three given eigenvectors are pairwise orthogonal, any non-zero vector perpendicular to them is automatically a fourth eigenvector (and v_4 above is in fact such a vector). On the other hand, v_4 doesn't have to be orthogonal to the three given eigenvectors: we could have chosen any vector $c_1(0,-1,1,0) + c_2(1,-1,-1,-1)$ with $c_2 \neq 0$.

d) Find the solution to $du/dt = Hu$ given that $u(0) =$ third column of S.

Solution: Let v_3 be the third column of S. It is an eigenvector corresponding to $\lambda_3 = -2$, so $u = e^{-2t}v_3$ is a solution to $du/dt = Hu$, and in fact it gives $u(0) = v_3$, so it is the desired solution.

3 (30 pts.) Suppose A is a 3 by 3 symmetric matrix with eigenvalues $2, 5, 7$ and corresponding eigenvectors x_1, x_2, x_3.

a) Suppose x is a combination $x = c_1x_1 + c_2x_2 + c_3x_3$. *Find* Ax. Now find $x^{\mathrm{T}}Ax$ using the symmetry of A. Prove that $x^{\mathrm{T}}Ax > 0$ (unless $x = 0$).

Solution: We write

$$Ax = c_1Ax_1 + c_2Ax_2 + c_3Ax_3 = 2c_1x_1 + 5c_2x_2 + 7c_3x_3$$

and

$$x^TAx = (c_1x_1^T + c_2x_2^T + c_3x_3^T)(2c_1x_1 + 5c_2x_2 + 7c_3x_3) =$$
$$= 2c_1^2x_1^Tx_1 + 5c_2^2x_2^Tx_2 + 7c_3^2x_3^Tx_3$$

(opening the parentheses, we use the fact that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal, and hence $x_i^Tx_j = 0$ for $i \neq j$). Since $x_i^Tx_i = ||x_i||^2 > 0$ and $c_i^2 > 0$ unless $c_i = 0$, we conclude that $x^TAx > 0$ unless $c_1 = c_2 = c_3 = 0$, i.e. $x = 0$.

b) Suppose those eigenvectors have length 1 (unit vectors). Show that $B = 2x_1x_1^{\mathrm{T}} + 5x_2x_2^{\mathrm{T}} + 7x_3x_3^{\mathrm{T}}$ has the same eigenvectors and eigenvalues as A. Is B necessarily the same matrix as A (yes or no)?

Solution: We have

$$Bx_1 = 2x_1x_1^Tx_1 + 5x_2x_2^Tx_1 + 7x_3x_3^Tx_1 = 2x_1$$

because $x_1^Tx_1 = ||x_1||^2 = 1$ and $x_i^Tx_j = 0$ for $i \neq j$. Thus x_1 is an eigenvector of B with eigenvalue $\lambda_1 = 2$. Similarly, we can show that $Bx_2 = 5x_2$ and $Bx_3 = 7x_3$. Since both A and B have diagonalization

$$\begin{bmatrix} & & \\ x_1 & x_2 & x_3 \\ & & \end{bmatrix}\begin{bmatrix} 2 & & \\ & 5 & \\ & & 7 \end{bmatrix}\begin{bmatrix} & & \\ x_1 & x_2 & x_3 \\ & & \end{bmatrix}^{-1},$$

they are the same matrix.

c) For which numbers b does this matrix have 3 positive eigenvalues?

$$A = \begin{bmatrix} 2 & b & 3 \\ b & 2 & b \\ 3 & b & 4 \end{bmatrix}$$

Solution: A has 3 positive eigenvalues if and only if it is positive-definite. To test for positive-definiteness, we check the three upper-left determinants to see when they are positive. The 1 by 1 upper-left determinant is 2, which is positive. The 2 by 2 upper-left determinant is $4 - b^2$, which is positive whenever $-2 < b < 2$. Finally, we compute the 3 by 3 upper-left determinant, or $\det A$:

$$\det A = 2 \det \begin{bmatrix} 2 & b \\ b & 4 \end{bmatrix} - b \det \begin{bmatrix} b & b \\ 3 & 4 \end{bmatrix} + 3 \det \begin{bmatrix} b & 2 \\ 3 & b \end{bmatrix} =$$

$$= 2(8 - b^2) - b(4b - 3b) + 3(b^2 - 6) = -2,$$

which is always negative. Since $\det A < 0$ regardless of the value of b, we conclude that A cannot have 3 positive eigenvalues.

Note: The SVD will be on the final when you have more time to digest it.

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