18.06 Professor Strang Quiz 1 October 5, 2005

Your PRINTED name is: SOLUTIONS

Grading

1

2

3

Please circle your recitation:

1)	M 2	2-131	P. Lee	2-087	2-1193	`lee`
2)	M 2	2-132	T. Lawson	4-182	8-6895	`tlawson`
4)	T 10	2-132	P.-O. Persson	2-363A	3-4989	`persson`
5)	T 11	2-131	P.-O. Persson	2-363A	3-4989	`persson`
6)	T 11	2-132	P. Pylyavskyy	2-333	3-7826	`pasha`
7)	T 12	2-132	T. Lawson	4-182	8-6895	`tlawson`
8)	T 12	2-131	P. Pylyavskyy	2-333	3-7826	`pasha`
9)	T 1	2-132	A. Chan	2-588	3-4110	`alicec`
10)	T 1	2-131	D. Chebikin	2-333	3-7826	`chebikin`
11)	T 2	2-132	A. Chan	2-588	3-4110	`alicec`
12)	T 3	2-132	T. Lawson	4-182	8-6895	`tlawson`

1 (30 pts.) Suppose A is m by n with **linearly dependent columns**. Complete with as much true information as possible:

(a) The rank of A is ...

at most $n-1$ (and at most m, which is a stronger statement if $m < n-1$).

(b) The nullspace of A contains ...

at least one non-zero vector. (The dimension of the nullspace is n minus the column rank of A, i.e., at least 1.)

(c) (more words needed) The equation $A^{\mathrm{T}}y = b$ has no solution for some right hand sides b because ...

the rows of the matrix A^{T}, which are the same as the columns of A, are linearly dependent, so A^{T} is not full row-rank. Thus the reduced row echelon form of A^{T} contains a row of all zeroes, so the components of b must satisfy a certain linear relation in order for $A^{\mathrm{T}}y = b$ to have a solution.

2 (40 pts.) Suppose A is this 3 by 4 matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

(a) A specific basis for the column space of A is ____________.

(b) For which vectors $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ does $Ax = b$ have a solution? Give conditions on b_1, b_2, b_3.

(c) There is no 4 by 3 matrix B for which $AB = I$ (3 by 3). Give a good reason (is this because A is rectangular?).

(d) Find the complete solution to $Ax = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Solution

(a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ is a basis for the column space of A. So is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b) Row reducing the augmented matrix $[\,A \;\; b\,]$, we get

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & b_1 \\ 0 & -1 & -2 & -3 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{array}\right].$$

The linear equations corresponding to the top two rows can be satisfied regardless of the values of b_1 and b_2, and the bottom row of all zeroes imposes the condition $b_3 - 2b_2 + b_1 = 0$. Hence $Ax = b$ has a solution if and only if $b_3 - 2b_2 + b_1 = 0$.

(c) This is because A is not full row-rank, as shown in part (b). If $AB = I$ then we could solve $Ab_1 =$ row 1 of I, $Ab_2 =$ row 2 of I, $Ab_3 =$ row 3 of I, and every equation $Ax = b$. Actually the solution would be $x = Bb$. But in part (b) we saw that $Ax = b$ has no solution for some b.

(d) We perform the row reduction

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 3 & 4 & 5 & 0 \\ 3 & 4 & 5 & 6 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & -1 & -2 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Then $x_p = \begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution, and $s_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $s_4 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ are **special solutions** forming a basis of the nullspace of A. Hence the general solution is

$$x = x_p + x_n = x_p + cs_3 + ds_4 \,.$$

3 (30 pts.) (a) Find a basis for the vector space of all real 3 by 3 symmetric matrices.

(b) Suppose A is a square invertible matrix. You permute its rows by a permutation P to get a new matrix B. How do you know that B is also invertible?

(c) "If 2 matrices have the same shape and the same nullspace, then they have the same column space." **If this is true**, give a reason why. **If this is not true**, find 2 matrices to show it's false.

Solution

(a) The most natural basis is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(b) A being invertible means that A has full rank. Permuting the rows has no effect on the rank, so B has full rank as well, and is thus invertible. (Another argument: the permutation matrix P is invertible, and so $B^{-1} = (PA)^{-1} = A^{-1}P^{-1}$.)

(c) The statement is false. Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ have the same nullspace (the line spanned by the vector $(1, -1)$), but their column spaces differ (for A, it's the line spanned by the vector $(1, 1)$, and for B, it's the line spanned by the vector $(1, 2)$).

Remark (now on the web page)

The real 3 by 3 matrices form a vector space M. The symmetric matrices in M form a subspace S. If you add two symmetric matrices, or multiply by real numbers, the result is still a symmetric matrix. **Problem: Find a basis for $\boldsymbol{S}$.**

When I asked this question on an exam, I realized that a key point needs to be emphasized: **The basis "vectors" for $\boldsymbol{S}$ must lie in the subspace.** They are 3 by 3 symmetric matrices! Then there are two requirements:

1. The basis vectors must be linearly independent.
2. Their combinations must produce every vector (matrix) in S.

Here is one possible basis (all symmetric) for this example:

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad S_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad S_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad S_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Since this basis contains 6 vectors, the **dimension of $\boldsymbol{S}$ is 6.**