18.06 Problem Set 9

Due at 4pm on Wednesday, November 23 in 2-106

Please PRINT your name and recitation information on your homework

1. Section 6.4, Problem 4
2. Section 6.4, Problem 6
3. Section 6.4, Problem 10
4. Section 6.4, Problem 14
5. Section 6.4, Problem 25
6. Section 6.4, Problem 27
7. Section 6.5, Problem 7
8. Section 6.5, Problem 9
9. Section 6.5, Problem 15
10. Section 6.5, Problem 20
11. Section 6.5, Problem 28
12. Section 10.2, Problem 14
13. Section 10.2, Problem 16
14. In this problem you will find the matrix exponential e^{At}, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$

in three ways.

(a) Compute the diagonalization $A = S\Lambda S^{-1}$ and use $e^{At} = Se^{\Lambda t}S^{-1}$.

(b) Find two independent solutions to the system of differential equations $u' = Au$, where $u(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$; call them u_1 and u_2. Let $\Phi(t)$ be the 2 by 2 matrix having u_1 and u_2 as columns. Then the general solution to the system is $\Phi(t) \cdot \vec{c}$, where $\vec{c}$ is a vector of constants. The idea is to normalize $\Phi(t)$ by multiplying it by a constant matrix B such that $\Phi_N(t) = \Phi(t) \cdot B$ satisfies $\Phi_N(0) = I$. Find such a matrix B (with constant entries!). Then the general solution to the system becomes $u = \Phi_N(t) \cdot u(0)$, and hence $\Phi_N(t)$ is precisely the exponential e^{At}.

(c) Finally, e^{At} can be obtained directly from the power series definition of matrix exponential. It is not very convenient to deal with powers of At in this case. However, recall that if square matrices M and N commute, i.e. if $MN = NM$, then $e^M e^N = e^{M+N}$. Find a decomposition $At = Bt + cIt$ such that the powers of Bt are easily computable, and then find the exponential e^{Bt} directly from the corresponding power series. Then compute $e^{At} = e^{Bt}e^{cIt}$. (Note that the formula is applicable since cIt is a multiple of identity and hence commutes with everything.)

(d) Plug in $t = 1$ and $t = 2$ into e^{At}. How do these two matrices relate? Check your answer by direct calculation.