# 18.06 Problem Set 9

SOLUTIONS

1. Section 6.4, Problem 4

*Answer:* $Q = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

2. Section 6.4, Problem 6

*Answer:* $Q = \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$ or $Q = \begin{bmatrix} -.8 & .6 \\ .6 & .8 \end{bmatrix}$ or exchange columns.

3. Section 6.4, Problem 10

*Answer:* If $x$ is not real then $\lambda = x^T Ax/x^T x$ is not necessarily real. Can't assume real eigenvectors!

4. Section 6.4, Problem 14

*Answers:* Skew-symmetric and orthogonal; $\lambda = i, i, -i, -i$ to have trace 0.

5. Section 6.4, Problem 25

*Answers:* Symmetry gives $Q\Lambda Q^T$ when $b = 1$; repeated $\lambda$ and no $S$ when $b = -1$; singular if $b = 0$.

6. Section 6.4, Problem 27

*Answer:* Eigenvectors $(1, 0)$ and $(1, 1)$ give a 45 degree angle even with $A^T$ very close to $A$.

7. Section 6.5, Problem 7

*Answer:* $A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$ and $A^T A = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$ are positive definite; $A^T A = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ is singular.

8. Section 6.5, Problem 9

*Answers:* $A = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}$ has only one pivot $= 4$; rank of $A$ is 1; eigenvalues of $A$ are $24, 0, 0$; $\det A = 0$.

9. Section 6.5, Problem 15

*Solution:* Since $x^T Ax > 0$ and $x^T Bx > 0$, we have $x^T(A+B) = x = x^T Ax + x^T Bx > 0$ for all $x \neq 0$. Hence $A+B$ is a positive definite matrix.

10. Section 6.5, Problem 20

*Answers:* (a) The determinant is positive, all $\lambda > 0$;

(b) All projection matrices except $I$ are singular (because for all such matrices there is a non-zero vector whose projection is 0);

(c) The diagonal entries of $D$ are its eigenvalues;

(d) $-I$ has det $= 1$ when $n$ is even.

11. Section 6.5, Problem 28

*Answers:* $\det A = 10$; $\lambda = 2, 5$; $x_1 = (\cos\theta, \sin\theta)$, $x_2 = (-\sin\theta, \cos\theta)$; the $\lambda$'s are positive.

12. Section 10.2, Problem 14

*Solution:* If $U^H U = I$ then $U^{-1}(U^H)^{-1} = U^{-1}(U^{-1})^H = I$ so $U^{-1}$ is also unitary. Also $(UV)^H UV = V^H U^H UV = V^H V = I$, so $UV$ is unitary.

13. Section 10.2, Problem 16

*Solution:* $(z^H A^H)(Az) = ||Az||^2$ is positive unless $Az = 0$; with independent columns $Az = 0$ means $z = 0$. Thus $z^H A^H Az > 0$ for $z \neq 0$, and $A^H A$ is positive definite.

14. (a) The eigenvalues of $A$ are $\lambda_{1,2} = 3, -1$, and the corresponding eigenvectors are $v_1 = (1, 1)$ and $v_2 = (1, -1)$. Thus we have

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & \\ & -1 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and

$$e^{At} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{3t} & \\ & e^{-t} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{3t} + e^{-t} & e^{3t} - e^{-t} \\ e^{3t} - e^{-t} & e^{3t} + e^{-t} \end{bmatrix}.$$

(b) From the eigenvalues and the eigenvectors of $A$ obtained in the previous part we get two independent solutions to $u' = Au$: $u_1 = e^{3t} v_1$ and $u_2 = e^{-t} v_2$. Thus

$$\Phi(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}$$

is a fundamental matrix. Normalizing, we get

$$e^{At} = \Phi_N(t) = \Phi(t) \cdot (\Phi(0))^{-1} = \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \cdot \left( \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^{-1} = \frac{1}{2} \begin{bmatrix} e^{3t} + e^{-t} & e^{3t} - e^{-t} \\ e^{3t} - e^{-t} & e^{3t} + e^{-t} \end{bmatrix}.$$

(c) Decompose $At = Bt - cIt = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} t - It$. We have

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}^2 = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}; \quad \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}^3 = \begin{bmatrix} 32 & 32 \\ 32 & 32 \end{bmatrix}; \quad \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}^k = \frac{1}{2}\begin{bmatrix} 4^k & 4^k \\ 4^k & 4^k \end{bmatrix}.$$

Hence

$$e^{Bt} = I + \frac{1}{2}\sum_{k=1}^{\infty} \frac{t^k}{k!}\begin{bmatrix} 4^k & 4^k \\ 4^k & 4^k \end{bmatrix} = I + \frac{1}{2}\begin{bmatrix} e^{4t}-1 & e^{4t}-1 \\ e^{4t}-1 & e^{4t}-1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{4t}+1 & e^{4t}-1 \\ e^{4t}-1 & e^{4t}+1 \end{bmatrix}$$

and

$$e^{At} = e^{Bt} \cdot e^{-It} = \frac{1}{2}\begin{bmatrix} e^{4t}+1 & e^{4t}-1 \\ e^{4t}-1 & e^{4t}+1 \end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{-t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{3t}+e^{-t} & e^{3t}-e^{-t} \\ e^{3t}-e^{-t} & e^{3t}+e^{-t} \end{bmatrix}.$$

(d) Plugging in $t = 1$ and $t = 2$, we have

$$e^A = \frac{1}{2}\begin{bmatrix} e^3+e^{-1} & e^3-e^{-1} \\ e^3-e^{-1} & e^3+e^{-1} \end{bmatrix}; e^{2A} = \frac{1}{2}\begin{bmatrix} e^6+e^{-2} & e^6-e^{-2} \\ e^6-e^{-2} & e^6+e^{-2} \end{bmatrix}.$$

The second matrix is the square of the first, as can be checked by direct computation.