18.06 Problem Set 8

SOLUTIONS

1. Section 6.2, Problem 42

Answers: B has eigenvalues $\lambda = i, -i$, so B^4 has eigenvalues $\lambda^4 = 1, 1$; C has eigenvalues $\lambda = (1 \pm i\sqrt{3})/2 = \exp(\pm\pi i/3)$, so C^3 has eigenvalues $\lambda^3 = -1, -1$. Thus $C^3 = -I$ and $C^{1024} = -C$.

2. Section 6.3, Problem 3

Answers: $\begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$, and $\lambda = \frac{1}{2}(5 \pm \sqrt{41})$.

3. Section 6.3, Problem 10

Solution: (b) When A is skew-symmetric, $||u(t)|| = ||e^{At}u(0)|| = ||u(0)||$. So e^{At} is an *orthogonal* matrix. Another argument:

$$Q = e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!},$$

so

$$Q^T = \sum_{k=0}^{\infty} \frac{(A^T)^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{(-A)^k t^k}{k!} = e^{-At}.$$

Hence $Q^T Q = e^{-At} e^{At} = I$.

4. Section 6.3, Problem 11

Answers: (a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

(b) $u(t) = \frac{1}{2} e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2} e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$.

5. Section 6.3, Problem 23

Solution: We have $A^2 = A$, so $A^3 = A^4 = \cdots = A$, and

$$e^{At} = I + \sum_{k=1}^{\infty} \frac{A^k t^k}{k!} = I + \sum_{k=1}^{\infty} \frac{A t^k}{k!} = I + \left(\sum_{k=1}^{\infty} \frac{t^k}{k!} \right) \cdot A = I + (e^t - 1)A = \begin{bmatrix} e^t & 3(e^t - 1) \\ 0 & 1 \end{bmatrix}.$$

6. Section 8.3, Problem 7

Answers: $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & .5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix}$; A^{16} has the same factors with $(.5)^{16}$ instead of .5 in the middle matrix.

7. Section 8.3, Problem 8

Answers: $(.5)^k \to 0$ gives $A^k \to A^\infty$; the general form of a matrix producing the steady state $(.6, .4)$ is $A = \begin{bmatrix} .6 + .4a & .6 - .6a \\ .4 - .4a & .4 + .6a \end{bmatrix}$ with $-2/3 \leq a \leq 1$.

8. Section 8.3, Problem 9

Answers: $u_1 = (0,0,1,0)$, $u_2 = (0,1,0,0)$, $u_3 = (1,0,0,0)$, $u_4 = u_0 = (0,0,0,1)$. The eigenvalues are all the fourth roots of 1, namely, 1, i, -1, $-i$. As demonstrated by this example, a Markov matrix with zeroes may have more than one eigenvalue of magnitude 1; a Markov matrix containing only positive entries has exactly one eigenvalue of magnitude 1 (and equal to 1).

9. Section 8.3, Problem 10

Solution: Suppose M is a Markov matrix. Since the entries of M are nonnegative, so are the entries of M^2. The columns of M each add up to 1, i.e. $[1\ 1\ \ldots\ 1]M = [1\ 1\ \ldots\ 1]$, hence

$$[1\ 1\ \ldots\ 1]M^2 = ([1\ 1\ \ldots\ 1]M) \cdot M = [1\ 1\ \ldots\ 1]M = [1\ 1\ \ldots\ 1],$$

i.e. the columns of M^2 each add up to 1. We conclude that M^2 is also a Markov matrix.

10. Section 8.3, Problem 12

Solution: To make the columns add up to 1, the last row has to be $.2, .3, .5$. If A is a symmetric Markov matrix, then not only its columns but also its rows each add up to 1, and hence

$$A \cdot \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix},$$

i.e. $(1, \ldots, 1)$ is the steady state eigenvector of A.

11. (a) After reordering the eigenvalues and scaling the eigenvectors, we obtain the following

results:

$$A_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{bmatrix} \begin{bmatrix} -0.4 & 0.5 & 0 & 0.1 \\ 0.5 & -1 & 0.5 & 0 \\ 0 & 0.5 & -1 & 0.5 \\ 0.1 & 0 & 0.5 & -0.4 \end{bmatrix};$$

$$A_5 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 2 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 5 \end{bmatrix} \cdot \frac{1}{12} \begin{bmatrix} -5 & 6 & 0 & 0 & 1 \\ 6 & -12 & 6 & 0 & 0 \\ 0 & 6 & -12 & 6 & 0 \\ 0 & 0 & 6 & -12 & 6 \\ 1 & 0 & 0 & 6 & -5 \end{bmatrix};$$

$$A_6 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 & 2 & 3 \\ 5 & 4 & 3 & 2 & 1 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ & & 3 & & & \\ & & & 4 & & \\ & & & & 5 & \\ & & & & & 6 \end{bmatrix} \cdot \frac{1}{14} \begin{bmatrix} -6 & 7 & 0 & 0 & 0 & 1 \\ 7 & -14 & 7 & 0 & 0 & 0 \\ 0 & 7 & -14 & 7 & 0 & 0 \\ 0 & 0 & 7 & -14 & 7 & 0 \\ 0 & 0 & 0 & 7 & -14 & 7 \\ 1 & 0 & 0 & 0 & 7 & -6 \end{bmatrix}.$$

(b) The data from part (a) suggest that

$$v_k = \begin{bmatrix} k \\ k-1 \\ k-2 \\ \dots \\ 1 \\ 2 \\ 3 \\ \dots \\ n-k+1 \end{bmatrix}$$

is an eigenvector of A_n with eigenvalue k. Let us prove it formally by carrying out the multiplicatin $A_n v_k$. Let $r_n^{(i)}$ denote the i-th row of n. Then

$$r_n^{(1)} \cdot v_k = \frac{1}{n+1}\left(\frac{k(n+1)(n+2)}{2} + k + ((k-1)+(k-2)+\cdots+1)(n+1) + \right.$$

$$\left. + (2+3+\cdots+(n-k))(n+1) - \frac{n(n+1)(n+1-k)}{2} + n + 1 - k\right) =$$

$$= \frac{1}{n+1}\left(\frac{k(n+1)(n+2)}{2} + \frac{k(k-1)(n+1)}{2} + \right.$$

$$+\frac{(n-k-1)(n-k+2)(n+1)}{2}-\frac{n(n+1)(n+1-k)}{2}+n+1\Big)=$$

$$=\frac{1}{n+1}\left(\frac{n+1}{2}\cdot(k(n+2)+k(k-1)+(n-k-1)(n-k+2)-n(n+1-k)+2)\right)=$$

$$=\frac{1}{2}\cdot 2k^2=k^2,$$

which is k times the first entry of v_k, as needed. Then

$$r_n^{(2)}\cdot v_k-r^{(1)}\cdot v_k=\frac{1}{n+1}((-1-2n)k+(n+1)v_k(2)+n+1-k)=v_k(2)-2k+1,$$

where $v_k(2)$ is the second entry of v_k (we simply examine the differences between the corresponding entries in the first and in the second row). For $k=1$, this difference is 1, hence $r_n^{(2)}\cdot v_k=2$, and for $k>1$, this difference is $k-1-2k+1=-k$, hence $r_n^{(2)}\cdot v_k=k^2-k$, as needed. For $3\leq i\leq n-1$, we have

$$r_n^{(i)}\cdot v_k-r_n^{(i-1)}\cdot v_k=\frac{1}{n+1}(k-i(n+1)v_k(i-1)+(i-1)(n+1)v_k(i)+n+1-k)=$$

$$=i(v_k(i)-v_k(i-1))-v_k(i)+1.$$

If $i\leq k$, then we have $v_k(i)=k-i+1$ and $v_k(i-1)=k-i+2$, so the above difference equals $-k$. If $i>k$, then $v_k(i)=i-k+1$ and $v_k(i-1)=i-k$, so the above difference equals k. Hence

$$r^{(1)}\cdot v_k=k^2;$$
$$r^{(2)}\cdot v_k=k^2-k;$$
$$r^{(3)}\cdot v_k=k^2-2k;$$
$$\cdots$$
$$r^{(k)}\cdot v_k=k;$$
$$r^{(k+1)}\cdot v_k=2k;$$
$$\cdots$$
$$r^{(n-1)}\cdot v_k=(n-k)k.$$

A calculation similar to the computation of the product $r^{(1)}\cdot v_k$ shows that $r^{(n)}\cdot v_k=(n+1-k)k$. Hence v_k is indeed an eigenvector of A_n with eigenvalue k.