18.06 Problem Set 6

SOLUTIONS

1. Section 4.4, Problem 20

Answers: (a) True. (b) True. $Qx = x_1q_1 + x_2q_2$. $||Qx||^2 = x_1^2 + x_2^2$ because $q_1 \cdot q_2 = 0$.

2. Section 4.4, Problem 21

Answers: The orthonormal vectors are $q_1 = (1,1,1,1)/2$ and $q_2 = (-5,-1,1,5)/\sqrt{52}$. Then $b = (-4,-3,3,0)$ projects to $p = (-7,-3,-1,3)/2$.

3. Section 4.4, Problem 23

Answers: $q_1 = (1,0,0)$, $q_2 = (0,0,1)$, $q_3 = (0,1,0)$. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.

4. Section 4.4, Problem 34

Solution: (a) $Qu = (I - 2uu^T)u = u - 2uu^Tu$. This is $-u$, provided that $u^Tu = 1$, i.e. u is a unit vector.

(b) $Qv = (I - 2uu^T)v = u - 2uu^Tv = u$, provided that $u^Tv = 0$.

6. Section 5.1, Problem 3

Answers: (a) False (2 by 2 identity matrix)

(b) True

(c) False (2 by 2 identity matrix)

(d) False (but trace is 0)

7. Section 5.1, Problem 12

Solution: If an n by n matrix is multiplied by a constant k, then the determinant is multiplied by k^n, not k. Thus the numerator in the given calculation should be $(ad - bc)^2$, and the correct determinant of A^{-1} is $1/(ad - bc)$.

8. Section 5.1, Problem 13

Answers: Pivots 1, 1, 1 give det = 1; pivots 1, -2, $-3/2$ give det = 3.

9. Section 5.1, Problem 18

Solution:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

10. Section 5.1, Problem 19

Answers: $\det U = 6$, $\det U^{-1} = 1/6$, $\det U^2 = 36$; $\det U = ad$, $\det U^2 = a^2d^2$, $\det U^{-1} = 1/ad$ (unless $ad = 0$, in which case no U^{-1} exists).

11. Section 5.1, Problem 28

Solution: (a) True: $\det AB = \det A \det B = 0$.

(b) False: may exchange rows.

(c) False: $A = 2I$, $B = I$.

(d) True: $\det AB = \det A \det B = \det BA$.

12. Section 5.1, Problem 34

Answer: $\det B = -6$.

13. Section 5.2, Problem 5

Answers: $\det A = a_{11}a_{23}a_{32}a_{44} + a_{14}a_{23}a_{32}a_{41} = 0$; $\det B = 2 \cdot 4 \cdot 4 \cdot 2 - 1 \cdot 4 \cdot 4 \cdot 1 = 48$.

14. Section 5.2, Problem 17

Solution: The $(1,1)$ cofactor is E_{n-1}, and the $(1,2)$ cofactor has a single 1 in its first column, with cofactor E_{n-2}. Hence $E_n = E_{n-1} - E_{n-2}$. The sequence E_1, E_2, $\ldots$ is then the periodic sequence 1, 0, -1, -1, 0, 1, $\ldots$ with period 6. Therefore $E_{100} = -1$.

15. Section 5.2, Problem 21

Answer: $G_2 = -1$, $G_3 = 2$, $G_4 = -3$. In general, $G_n = (-1)^{n-1}(n-1)$.

16. Section 5.2, Problem 25

Solution: (a) Computing the determinant using the big formula, the terms involving an entry of B, i.e. an entry from the first two rows and the last two columns, would also have to involve an entry from the last two rows and the first two columns, and all such entries are equal to 0. Hence all terms involving entries of B are equal to 0 in this case. The remaining terms are all possible products of a term from the big formula for A and a term of the big formula for D (the signs also multiply!), so the determinant of the block matrix is $\det A \det D$.

(b) and (c) Here is a counterexample to both parts:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

17. Section 5.3, Problem 6

Answers: (a) $\begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & -4/3 & 1 \end{bmatrix}$

(b) $\frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. The inverse of a symmetric matrix is symmetric.

18. Section 5.3, Problem 7

Solution: If all cofactors were 0 and A were invertible, then A^{-1} would have to be the zero matrix, which is absurd, as A would then have to be the inverse of the zero matrix. The matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ does not have cofactors equal to 0, yet A is not invertible.