18.06 Problem Set 3

SOLUTIONS

1. Section 2.7, Problem 24

Answer: $\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ & 3 & 8 \\ & & -2/3 \end{bmatrix}$;
$A = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

2. Section 3.1, Problem 4

Answer: The zero vector is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $\frac{1}{2}A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$; $-A = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$.
The smallest subspace containing A consists of all matrices of the form cA, where c is an arbitrary number.

3. Section 3.1, Problem 19

Answer: The column space of A is the x-axis, i.e. all vectors of the from $(x, 0, 0)$. The column space of B is the xy-plane, i.e. all vectors of the form $(x, y, 0)$. The column space of C is the line of vectors $(x, 2x, 0)$.

4. Section 3.1, Problem 27

Answer: (a) False; (b) True; (c) True; (d) False.

5. Section 3.2, Problem 23

Answer: $A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$.

6. Section 3.2, Problem 25

Answer: $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$.

7. Section 3.3, Problem 3

Answer: $R_A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$; $R_B = [R_A \; R_A]$; R_C is the matrix $\begin{bmatrix} R_A & 0 \\ 0 & R_A \end{bmatrix}$ with the row of all zeroes from R_A moved all the way to the bottom.

8. Section 3.3, Problem 6

Answer: A and A^T have the same rank r. But *pivcol* (the column number) is 2 for A and 1 for A^T since $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

9. Section 3.3, Problem 8

Answer: $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}$; $B = \begin{bmatrix} 2 & 6 & -3 \\ 1 & 3 & -3/2 \\ 2 & 6 & -3 \end{bmatrix}$; $M = \begin{bmatrix} a & b \\ c & bc/a \end{bmatrix}$.

10. Section 3.4, Problem 5

Answer: Solvable if $2b_1 + b_2 = b_3$. Then $x = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

11. Section 3.4, Problem 32

Answer: $A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $x = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix} + c \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}$. No solution for $b = (1, 0, 0, 0)$.

12. Section 3.4, Problem 33

Answer: $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$.

13. Section 3.4, Problem 34

Solution: (a) The dimension of the nullspace of A is 1, so the dimension of the column space is 3, hence the matrix A has rank 3. The complete solution to $Ax = 0$ is $x = c\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$.

(b) $R = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

14. (a) B is n by m, and I is m by m.

(b) `A\I` gives the matrix $\begin{bmatrix} 0 & 1/4 \\ 0 & 0 \\ 1/2 & 5/8 \end{bmatrix}$.

(c) `rref([A I])` yields the matrix $\begin{bmatrix} 1 & 0 & 2 & 1 & 3/2 \\ 0 & 1 & 4 & 2 & 5/2 \end{bmatrix}$. Set B to be the matrix of unknowns

$$B = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}.$$

We want to solve the system $AB = I$ for B. Then from the above reduction of the augmented matrix $[A\ I]$, we get

$$\begin{bmatrix} x_1 + 2x_3 & y_1 + 2y_3 \\ x_2 + 4x_3 & y_2 + 4y_3 \end{bmatrix} = \begin{bmatrix} 1 & 3/2 \\ 2 & 5/2 \end{bmatrix},$$

so setting $x_3 = y_3 = 0$ and $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ to be the matrix consisting of the last two columns of `rref([A I])` yields the following right inverse of A:

$$B = \begin{bmatrix} 1 & 3/2 \\ 2 & 5/2 \\ 0 & 0 \end{bmatrix}.$$

(d) Suppose $CA = I$, where C is an n by m matrix, and I is the n by n identity matrix. The matrices C and A each have rank at most m because the rank of a matrix cannot be greater than the number of rows or the number of columns. Recall that the rank of the product of matrices does not exceed the rank of either of the matrices being multiplied, so the rank of CA is at most m, whereas the rank of I is n. Since $m < n$, we conclude that $CA \neq I$, and A has no left inverse.