18.06 Fall 2004 Quiz 2 November 15, 2004

Your name is:

Please circle your recitation:

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4. T10 K. Cheung
5. T10 Y. Rubinstein
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7. T11 V. Angeltveit
8. T12 V. Angeltveit
9. T12 F. Rochon
10. T1 L. Williams
11. T1 K. Cheung
12. T2 T. Gerhardt

Grading:

Question	Points	Maximum
Name + rec		5
1		22
2		20
3		25
4		28
Total:		100

Remarks:
Do all your work on these pages.
No calculators or notes.
Putting your name and recitation section correctly is worth 5 points.
The exam is worth a total of 100 points.

1. (22 pts.)

(a) For the following 3×3 matrix A, compute its determinant by using the cofactor formula and expanding along the third column. Show the values of the 3 cofactors you compute.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -2 \\ 1 & -4 & 1 \end{bmatrix}.$$

Solution:

$$\begin{aligned} \det(A) &= 3(\det\left(\begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}\right)) - 2(-\det\left(\begin{bmatrix} 1 & 2 \\ 1 & -4 \end{bmatrix}\right)) + 1(\det\left(\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}\right)) \\ &= 3(4-2) - 2(-(-4-2)) + 1(2-(-2)) \\ &= 3(2) - 2(6) + 1(4) \\ &= -2. \end{aligned}$$

The cofactors are of course the terms in the brackets in the penultimate line.

(b) Consider the matrix B obtained from A by adding rows 1 and 3 to row 2. Should $\det(B)$ equal $\det(A)$? Why?

Solution: Yes. Adding a multiple of one row to another does not change the determinant.

Compute $\det(B)$ directly from B.

Solution:

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 1 & -4 & 1 \end{bmatrix}$$

We'll expand along the second row because it looks easiest.

$$\begin{aligned} \det(B) &= 1(-(2+12)) + 0(\text{whatever}) + 2(-(-4-2)) \\ &= -14 + 12 \\ &= -2 \end{aligned}$$

Sure enough, this is the same as the answer for part (a).

2. (20 pts.) Give all values of x for which A has an eigenvalue equal to 2.

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & x & 2 \\ x & -2 & 3 \end{bmatrix}.$$

Solution: The matrix A will have 2 as an eigenvalue if and only $\det(A - 2I) = 0$. Therefore, to achieve this, it is necessary and sufficient that x satisfy

$$\begin{aligned} 0 &= \det(A - 2I) \\ &= \det\left(\begin{bmatrix} 1 & 2 & -1 \\ 2 & x-2 & 2 \\ x & -2 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 1 & 2 & -1 \\ 2 & x-2 & 2 \\ x+1 & 0 & 0 \end{bmatrix}\right) \qquad \text{(after adding row 1 to row 3)} \\ &= (x+1)(4 + x - 2) \\ &= (x+1)(x+2). \end{aligned}$$

Hence, A has 2 as an eigenvalue if and only if x equals -1 or -2.

3. (25 pts.)

(a) Use the Gram-Schmidt procedure to find an **orthonormal** basis of $C(A)$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Solution:

$$u_1 = a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$u_2 = a_2 - \frac{a_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\begin{aligned} u_3 &= a_3 - \frac{a_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{a_3 \cdot u_2}{u_2 \cdot u_2} u_2 \\ &= \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} - \frac{-2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Now

$$q_1 = \frac{1}{\|u_1\|} u_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad q_2 = \frac{1}{\|u_2\|} u_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \qquad q_3 = \frac{1}{\|u_3\|} u_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}.$$

So an orthonormal basis for $C(A)$ is

$$\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} \right\}.$$

(b) Find the projection matrix P for projecting onto $C(A)$.

Solution: Recall that with the factorization $A = QR$, the projection matrix $P = A(A^TA)^{-1}A^T$ becomes QQ^T. Hence, we have

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(c) Check your answer for P by computing Pa_1 where a_1 is the first column of A.

Solution: We have

$$Pa_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = a_1$$

as expected.

4. (28 pts.) For each of the following statements, determine if it is *always* true. If so, answer yes. Otherwise, answer no. Just circle yes or no. You do not need to justify your answer. However, we will be giving 4 points for a correct answer, 0 points for no answer at all, and **-2 points for an incorrect answer.**

(a) **Yes — No.** Let F be the vector space of all 3×3 matrices. Let T be the transformation that maps $A \in F$ to $\det(A) + \text{trace}(A)$. Then T is a linear transformation.

Solution: No (Reason: $T(2I) \neq 2T(I)$, for example)

(b) **Yes — No.** Let A be the $m \times n$ (edge-node) incidence matrix of a connected graph with n vertices and m edges. Then the left nullspace of A has dimension $m - n + 1$.

Solution: Yes (Reason: the rank of A is $n - 1$.)

(c) **Yes — No.** An orthogonal matrix can have an eigenvalue equal to 0.

Solution: No (Reason: an orthogonal matrix is non-singular as the columns are perpendicular and so certainly independent.)

(d) **Yes — No** Let $\hat{x}$ be the least-squares solution to $Ax = b$. Then $b - A\hat{x}$ is orthogonal to the column space of A.

Solution: Yes (By definition!)

(e) **Yes — No.** Let P be the projection matrix for projecting onto a subspace F. Then $I - P$ is the projection matrix for projecting onto $F^{\perp}$.

Solution: Yes (Reason: the sum of the projections must be the identity)

(f) **Yes — No.** There are values for a, b, c, d, e, f such that the following matrix A satisfies $A^2 = 2A$:

$$A = \begin{bmatrix} 1 & a & b \\ c & 1 & d \\ e & f & 1 \end{bmatrix}.$$

Solution: No (Reason: multiplying by any eigenvector, and then considering a non-zero entry of it and dividing out, we get that *any* eigenvalue must satisfy $\lambda^2 = 2\lambda$, ie, $\lambda = 0$ or 2. But the sum of the eigenvalues of A is $\text{trace}(A) = 3$ which cannot be obtained by adding 0's and 2's.)

(g) **Yes — No.** Let 0 denote the 5×5 matrix whose entries are all 0. If $A^{10} = 0$ then $A = 0$.

Solution: No (Counterexample: $A = 0$ except for $A_{1,10} = 1$.)