18.06 Fall 2004 Quiz 1 October 13, 2004

Your name is:

Please circle your recitation:

1. M2 A. Brooke-Taylor
2. M2 F. Liu
3. M3 A. Brooke-Taylor
4. T10 K. Cheung
5. T10 Y. Rubinstein
6. T11 K. Cheung
7. T11 V. Angeltveit
8. T12 V. Angeltveit
9. T12 F. Rochon
10. T1 L. Williams
11. T1 K. Cheung
12. T2 T. Gerhardt

Grading:

Question	Points	Maximum
Name + rec		5
1		15
2		55
3		25
Total:		100

Remarks:
Do all your work on these pages.
No calculators or notes.
Putting your name and recitation section correctly is worth 5 points.
The exam is worth a total of 100 points.

1. Let

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 4 & 3 & 1 \\ -2 & -1 & 4 \end{bmatrix}.$$

(a) Compute an LDU factorization of A if one exists.

Solution:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{E_{31}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \underbrace{\begin{bmatrix} 2 & 2 & 2 \\ 4 & 3 & 1 \\ -2 & -1 & 4 \end{bmatrix}}_{A} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & -1 & -3 \\ 0 & 1 & 6 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{E_{32}} \begin{bmatrix} 2 & 2 & 2 \\ 0 & -1 & -3 \\ 0 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{bmatrix}$$

So for $A = LU$ decomposition, we have from this that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 2 & 2 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{bmatrix}$$

But this is not the U we want for $A = LDU$ decomposition; for that, we factor out the pivot values of the old U and put them in D. In this way we get

$$\underbrace{\begin{bmatrix} 2 & 2 & 2 \\ 4 & 3 & 1 \\ -2 & -1 & 4 \end{bmatrix}}_{A} \underset{=}{=} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}}_{U}.$$

(b) Give all solutions to $Ax = b$ where $b = \begin{bmatrix} 2 \\ -3 \\ 11 \end{bmatrix}$.

Solution: The quick way to do this is by forward and backward substitution, using the result of the previous part. We want $Ax = b$, that is, $LDUx = b$. Setting $DUx = c$, we have to first solve $Lc = b$ for c, and then $DUx = c$ for x.

Now, $Lc = b$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 11 \end{bmatrix}$$

so clearly we must have $c_1 = 2$, so $c_2 = -7$, and so $c_3 = 6$. With that, $DUx = c$ becomes

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ 6 \end{bmatrix},$$

and hence we get that $x_3 = 2$, so $x_2 = 1$, so finally $x_1 = -2$. Hence, the one and only solution to $Ax = b$ is $x = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$.

2. **One of the entries of A was modified as there was a mistake.** (Many of the subquestions are independent and can be answered in any order.) By performing row eliminations (and possibly permutations) on the following 4×8 matrix A

$$\begin{bmatrix} 1 & 2 & 0 & 3 & -1 & 1 & 1 & -2 \\ -3 & -6 & 2 & -7 & 7 & 0 & -6 & 3 \\ 1 & 2 & 2 & 5 & 3 & 3 & -1 & 0 \\ 2 & 4 & 0 & 6 & -2 & 1 & 3 & 0 \end{bmatrix}$$

we got the following matrix B:

$$\begin{bmatrix} 1 & 2 & 0 & 3 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) What is the rank of A?

Solution:
4. There are 4 pivots in the reduced row echelon form of A.

(b) What are the dimensions of the 4 fundamental subspaces?
Solution:

$$\begin{aligned} \dim(C(A)) &= \dim(R(A)) = \text{rank}(A) = 4 \\ \dim(N(A)) &= n - \text{rank}(A) = 8 - 4 = 4 \\ \dim(N(A^T)) &= m - \text{rank}(A) = 4 - 4 = 0 \end{aligned}$$

(c) How many solutions does $Ax = b$ have? Does it depend on b? Justify

Solution:
$Ax = b$ will have infinitely many solutions for any b. There is no row of 0's in the reduced row echelon form to cause there to be no solutions for the "wrong" b. There are infinitely many solutions since the nullspace, being 4-dimensional, has infinitely many elements.

(d) Are the rows of A linearly independent? Why?

Solution:
Yes. The reduced row echelon form of A has linearly independent rows, and row operations preserve the row space.

(e) Do columns 4, 5, 6 and 7 of A form a basis of R^4? Why?

Solution:
No. Columns 4, 5, 6 and 7 in B are dependent, and row operations preserve linear dependence and independence of columns. Hence, columns 4, 5, 6 and 7 of A are dependent.

(f) Give a basis of $N(A)$.

Solution: We get it as usual from the reduced row echelon form, B.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

gives the four equations

$$\begin{aligned} x_1 &= -2x_2 - 3x_4 + x_5 - 2x_7 \\ x_3 &= -x_4 - 2x_5 \\ x_6 &= x_7 \\ x_8 &= 0 \end{aligned}$$

From these we get the 4 special solutions corresponding to the 4 free variables x_2, x_4, x_5 and x_7. The special solutions are a basis for the nullspace. Hence, our basis is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(g) Give a basis of $N(A^T)$.

Solution:

We saw that $\dim(N(A^T)) = 0$. Hence, a basis for $N(A^T)$ must contain no vectors, that is, it must be the empty set $\{ \}$, often denoted by $\emptyset$.

(h) (You do not need to do any calculations to answer this question.) What is the reduced row echelon form for A^T? Explain.

Solution:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A^T is a 8×4 matrix with 4 independent columns (since A has 4 independent rows). Thus, every column in the reduced row echelon form must contain a pivot. Hence, the given matrix is the only possible reduced row echelon form of A^T.

(i) (Again calculations are not necessary for this part.) Let $B = EA$. Is E invertible? If so, what is the inverse of E?

Solution:

Yes; E is just the product of the elimination matrices (including possibly permutation matrices) which are applied to A to get B. Consider columns 1, 3, 6 and 8 of A - those which become the pivot columns of B. The matrix E is what performs this change on the columns. Hence,

$$E \begin{bmatrix} 1 & 0 & 1 & -2 \\ -3 & 2 & 0 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which of course means that this matrix is E^{-1}.

3. For each of these statements, say whether the claim is true or false and give a brief justification.

   (a) **True/False:** The set of 3×3 non-invertible matrices forms a subspace of the set of all 3×3 matrices.

   Solution:
   False. Consider for example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   The matrix on the right hand side is invertible, but the two on the left hand side are not.

   (b) **True/False:** If the system $Ax = b$ has no solution then A does not have full row rank.

   Solution:
   True. For $Ax = b$ to have no solution we must have a row of 0's in the reduced row echelon form. Hence, the number of pivots will be less than the number of rows, and so the matrix A does not have full rank.

(c) **True/False:** There exist $n \times n$ matrices A and B such that B is not invertible but AB is invertible.

Solution:
False. Suppose AB is invertible, and consider $C = (AB)^{-1}A$. Then

$$CB = (AB)^{-1}AB = I,$$

so C is an inverse for B.

(d) **True/False:** For any permutation matrix P, we have that $P^2 = I$.

Solution:
False. Consider the permutation matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then

$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \neq I.$$