18.06, Fall 2004, Problem Set 9

Due before 4PM on **Tuesday** November 23rd, 2004, in the boxes in 2-106. No late homework will be accepted. Don't forget to write your name, recitation section **and the names of students you have collaborated with** on the problem set. There is one box for each recitation section. For full credit, please be sure to show and explain your work. Exercises refer to the 3rd edition of the textbook.

Reading assignment: Sections 6.2, 8.3, 10.2

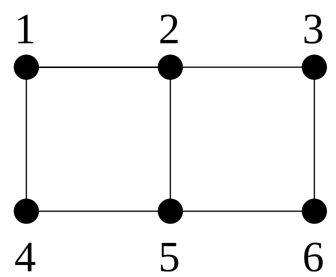
1. Let P be the projection matrix for projecting onto $C(A)$. Argue that P is diagonalizable.

2. Consider the matrix:

$$A = \begin{bmatrix} 0.5 & b & 0 & a \\ a & 0.5 & b & 0 \\ 0 & a & 0.5 & b \\ b & 0 & a & 0.5 \end{bmatrix}$$

   (a) For which values of a and b is A Markov?
   (b) Verify that $(1, 1, 1, 1)$ is an eigenvector of A. What is the corresponding eigenvalue? Is your result consistent with your answer to part 2a?
   (c) Let ω be a complex number such that $\omega^4 = 1$. What are all possible values for ω?
   (d) Verify that $(1, \omega, \omega^2, \omega^3)$ is an eigenvector whenever $\omega^4 = 1$. What are the corresponding eigenvalues?
   (e) For which values of a and b do we have an eigenvalue of algebraic multiplicity greater than 1. Explain.
   (f) Give a formula for the determinant of A in terms of a and b.
   (g) Can A be diagonalized for all values of a and b? [Do not assume that A is Markov.]
   (h) Give conditions on a and b such that A^k bounded for all k (i.e. the entries of A^k do not become arbitrarily large). [Do not assume that A is Markov.]
   (i) Verify your answers to (2d) and (2f) by using MATLAB. Take your MIT ID, and divide it by $2 \cdot 10^9$ to get a (so that a is between 0 and 0.5). Let $b = 0.5 - a$. Compute using MATLAB the eigenvalues of A and also compute $\det(A)$. Show your MATLAB output. Check your answer for (2d) and (2f).
   (j) For these same values of a and b, what does A^k tend to as k tends to infinity?

3. Consider the 2×3 grid shown below. Assume a mouse starts at vertex 1. At every step, the mouse either stays where it is with probability 0.5 or moves to an adjacent vertex selected uniformly among the current neighbors.



(a) What is the transition matrix A for this Markov Chain?

(b) What is the sum of the eigenvalues of A?

(c) Use MATLAB to compute the eigenvalues of A.

(d) In steady state, what will be the probability that the mouse is on one of the middle vertices 2 or 5?

4. Problem 16 on Page 493 in Section 10.2.