18.06, Fall 2004, Problem Set 7 Solutions

1. (11 pts.)

   (a) Let F be the subspace of all 2×2 matrices of the form:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

with $a+b+c+d=0$. It is clear that $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ and $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}$ are in F.
We claim that any element in F can be expressed as a linear combination of $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = -b\mathbf{a_1} - c\mathbf{a_2} - d\mathbf{a_3} = -b\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} - c\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} - d\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

since $a = -b-c-d$. So $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ span F.
Similarly, any element in F can be expressed as a linear combination of $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}$. In other words, we would like to find x_1, x_2, x_3 such that

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= x_1\mathbf{b_1} + x_2\mathbf{b_2} + x_3\mathbf{b_3} = x_1\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} + x_2\begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} + x_3\begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 - x_3 & -x_1 + 2x_2 \\ -x_1 - x_2 + 2x_3 & -x_2 - x_3 \end{bmatrix}. \end{aligned}$$

So we would like to be able to solve the system of equations:

$$\begin{cases} 2x_1 & & -x_3 & = a \\ -x_1 & +2x_2 & & = b \\ -x_1 & -x_2 & +2x_3 & = c \\ & -x_2 & -x_3 & = d \end{cases}$$

for every a, b, c, d with $a+b+c+d=0$. The last equation is implied by the first 3 (it is minus the sum of the first 3) as $a+b+c+d=0$, thus we just have to solve the system of equations:

$$\begin{cases} 2x_1 & & -x_3 & = a \\ -x_1 & +2x_2 & & = b \\ -x_1 & -x_2 & +2x_3 & = c \end{cases}$$

This system always has a (unique) solution since the underlying matrix

$$\begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

is nonsingular (e.g. its determinant is 5, or its columns are linearly independent, ...). This means that $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{b_3}$ also generate F.

(b) It is simpler to first compute the matrix $M = L^{-1}$ which allows to go from a representation $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{b_3}$ in the basis to a representation in the basis $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$. Indeed, the basis vectors $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{b_3}$ can be expressed in the basis $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ as

$$\mathbf{b_1} = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} = \mathbf{a_1} + \mathbf{a_2},$$

$$\mathbf{b_2} = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} = -2\mathbf{a_1} + \mathbf{a_2} + \mathbf{a_3},$$

$$\mathbf{b_3} = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} = -2\mathbf{a_2} + \mathbf{a_3}.$$

As the ith column of M corresponds to the coeffcients in the above expression of $\mathbf{b_i}$ we get:

$$M = L^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix}.$$

Now L is the inverse of this matrix which is (this can be obtained by Gauss-Jordan for example):

$$L = \frac{1}{5} \begin{bmatrix} 3 & 2 & 4 \\ -1 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix}.$$

(c) For

$$\mathbf{v} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix},$$

we get that $c_1 = 2$, $c_2 = 4$ and $c_3 = -3$. Computing

$$d = Lc = \frac{1}{5} \begin{bmatrix} 3 & 2 & 4 \\ -1 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 0.4 \\ -0.8 \\ -2.2 \end{bmatrix},$$

we verify that indeed

$$\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} = 0.4 \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} - 0.8 \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} - 2.2 \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$$

2. (11 pts.)

(a) For a permutation matrix P, the determinant can be either 1 (for any even permutation, such as the identity I), or -1 (for any odd permutation, such as an elementary permutation matrix). No other values are possible as any permutation matrix can be obtained from the identity by switching rows.

(b) For an orthogonal matrix Q, the determinant can be either 1 (e.g. for the identity) or -1 (e.g. for the identity after multiplying a row by -1). No other values are possible as $1 = \det(I) = \det(Q)\det(Q^T) = \det(Q)^2$.

(c) For a projection matrix P, the determinant can either be 1 (the only possibility here is the identity matrix corresponding to projecting on the entire space) or 0 (for any other projection matrix such as the 0 matrix corresponding to projecting over the subspace $\{0\}$). No other values are possible since $P^2 = P$ implies that $\det(P)^2 = \det(P)$ or $\det(P)(1-\det(P)) = 0$, implying that $\det(P) \in \{0, 1\}$.

(d) Computing explicitly the determinant of a 2×2 rotation matrix, we get $\cos^2(\theta)+\sin^2(\theta) = 1$. So 1 is the only value.

3. (7 pts.) We are going to use the facts that (i) when performing row eliminations or column eliminations, the determinant is unaffected and (ii) scaling a row or a column by t multiplies the determinant by t. Subtracting column 1 from columns 2, 3, 4, we get:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ a & b-a & c-a & d-a \\ a^2 & b^2-a^2 & c^2-a^2 & d^2-a^2 \\ a^3 & b^3-a^3 & c^3-a^3 & d^3-a^3 \end{vmatrix}.$$

Now, factorizing $(b-a)$ from column 2, $(c-a)$ from column 3 and $(d-a)$ from column 4, we get:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = (b-a)(c-a)(d-a)\begin{vmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 1 & 1 \\ a^2 & b+a & c+a & d+a \\ a^3 & b^2+ba+a^2 & c^2+ca+a^2 & d^2+da+a^2 \end{vmatrix}.$$

Substracting columns 2 from columns 3 and 4, we get:

$$\begin{aligned} &\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} \\ &= (b-a)(c-a)(d-a)\begin{vmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & b+a & c-b & d-b \\ a^3 & b^2+ba+a^2 & c^2-b^2+ca-ba & d^2-b^2+da-ba \end{vmatrix}. \end{aligned}$$

Factorizing $(c-b)$ from column 3 and $(d-b)$ from column 4, we get:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = (b-a)(c-a)(d-a)(c-b)(d-b)\begin{vmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & b+a & 1 & 1 \\ a^3 & b^2+ba+a^2 & a+b+c & a+b+d \end{vmatrix}.$$

Substracting column 3 from column 4, we get

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} &= (b-a)(c-a)(d-a)(c-b)(d-b)\begin{vmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & b+a & 1 & 0 \\ a^3 & b^2+ba+a^2 & a+b+c & d-c \end{vmatrix} \\ &= (b-a)(c-a)(d-a)(c-b)(d-b)(d-c) \\ &= (a-b)(a-c)(a-d)(b-c)(b-d)(c-d). \end{aligned}$$

4. (6 pts.) The determinant

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ x & 2 & 3 & 4 \\ 7 & 0 & 5 & 6 \\ 8 & 0 & 0 & 3 \end{vmatrix}$$

is a linear function of the form $ax+b$. Moreover, for $x=1$, we know it is equal to 0 as the first 2 rows are identical. Thus it is of the form $ax-a$. The rate of increase a of the determinant as x increases is given by the cofactor

$$C_{21} = -\begin{vmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 3 \end{vmatrix}.$$

As this is a diagonal matrix, we get that $C_{21} = -30$. Thus the determinant is equal to $-30x+30$ and this is equal to 10 for $x = \frac{2}{3}$.

5. (5 pts.) Assume by contradiction that there exists such a 7×7 matrix A which is both nonsingular and with the property that $A^T = -A$. Since it is nonsingular, we have $\det(A) \neq 0$. Moreover, from $A^T = -A$, we derive that $\det(A) = \det(A^T) = \det(-A) = (-1)^7 \det(A) = -\det(A)$, implying that $\det(A) = 0$, a contradiction. (Notice that if the size was even, say 6×6, then we would not derive such a contradiction as $\det(A^T) = \det(-A) = (-1)^6 \det(A) = \det(A)$.)