18.06, Fall 2004, Problem Set 6 Solutions

1. (12 pts.)

   (a) F is $N(A)$ where
   $$A = \begin{bmatrix} 1 & 2 & -1 & 1 \end{bmatrix}.$$
   To find a basis of F, we can just take the special solutions for $N(A)$ (the free variables are x_2, x_3 and x_4):
   $$\mathbf{a_1} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{a_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{a_3} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$
   The dimension of F is 3 (the dimension of $N(A)$).

   (b) We take
   $$\mathbf{u_1} = \mathbf{a_1} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$
   We take
   $$\mathbf{u_2} = \mathbf{a_2} - \frac{\mathbf{u_1}^T\mathbf{a_2}}{\mathbf{u_1}^T\mathbf{u_1}}\mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{-2}{5}\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \\ 1 \\ 0 \end{bmatrix}.$$
   (To make sure we haven't made any mistake we check that $\mathbf{u_2}$ indeed is in F and is orthogonal to $\mathbf{u_1}$. Yes.) We get
   $$\mathbf{u_3} = \mathbf{a_3} - \frac{\mathbf{u_1}^T\mathbf{a_3}}{\mathbf{u_1}^T\mathbf{u_1}}\mathbf{u_1} - \frac{\mathbf{u_2}^T\mathbf{a_3}}{\mathbf{u_2}^T\mathbf{u_2}}\mathbf{u_2} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{-1/5}{6/5}\begin{bmatrix} 1/5 \\ 2/5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/6 \\ -1/3 \\ 1/6 \\ 1 \end{bmatrix}.$$
   (Again we check that $\mathbf{u_3} \in F$ and that it is orthogonal to both $\mathbf{u_1}$ and $\mathbf{u_2}$.) Now we need scale these vectors to get unit vectors:
   $$\mathbf{q_1} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \\ 0 \end{bmatrix},$$
   $$\mathbf{q_2} = \frac{\sqrt{5}}{\sqrt{6}}\mathbf{u_2} = \begin{bmatrix} 1/5 \\ 2/5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ \sqrt{5}/\sqrt{6} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ 5/\sqrt{30} \\ 0 \end{bmatrix},$$

$$\mathbf{q_3} = \frac{\sqrt{6}}{\sqrt{7}}\begin{bmatrix} -1/6 \\ -1/3 \\ 1/6 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{42} \\ -2/\sqrt{42} \\ 1/\sqrt{42} \\ \sqrt{6}/\sqrt{7} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{42} \\ -2/\sqrt{42} \\ 1/\sqrt{42} \\ 6/\sqrt{42} \end{bmatrix}.$$

Again we check that $\mathbf{q_1}$, $\mathbf{q_2}$ and $\mathbf{q_3}$ are in F, are mutually orthogonal and have unit lengths.

(c) There are several ways of answering this question. We have to compute the length of $\mathbf{b} - \mathbf{p}$ where $\mathbf{b} = (3, 1, 1, 1)$ and $\mathbf{p}$ is the projection of $\mathbf{b}$ onto F.

The easiest way is to observe that $\mathbf{b} - \mathbf{p}$ is also equal to the projection of $\mathbf{b}$ onto the orthogonal complement of F. And $F^\perp$ is given by the line spanned by the vector $\mathbf{n} = (1, 2, -1, 1)$ (as $F = N(A)$). Thus the question is equivalent to asking what is the length of the projection $\mathbf{q}$ of $\mathbf{b}$ onto the line through $\mathbf{n} = (1, 2, -1, 1)$. We have

$$q = \frac{\mathbf{b}^T\mathbf{n}}{\mathbf{n}^T\mathbf{n}}\mathbf{n} = \frac{5}{7}\mathbf{n} = \frac{5}{7}\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}.$$

Thus the distance between $\mathbf{b}$ and F is given by $\frac{5}{7}\sqrt{7} = \frac{5}{\sqrt{7}}$.

Another way to compute $\mathbf{b} - \mathbf{p}$ is to notice that $\mathbf{b} - \mathbf{p}$ is the 4th vector we would get if we were to use Gram-Schmidt on $\mathbf{q_1}$, $\mathbf{q_2}$, $\mathbf{q_3}$ and $\mathbf{b}$. This gives

$$\begin{aligned}
\mathbf{b} - \mathbf{p} &= \mathbf{b} - \frac{\mathbf{q_1}^T\mathbf{b}}{\mathbf{q_1}^T\mathbf{q_1}}\mathbf{q_1} - \frac{\mathbf{q_2}^T\mathbf{b}}{\mathbf{q_2}^T\mathbf{q_2}}\mathbf{q_2} - \frac{\mathbf{q_3}^T\mathbf{b}}{\mathbf{q_2}^T\mathbf{q_3}}\mathbf{q_3} = \mathbf{b} - (\mathbf{q_1}^T\mathbf{b})\mathbf{q_1} - (\mathbf{q_2}^T\mathbf{b})\mathbf{q_2} - (\mathbf{q_3}^T\mathbf{b})\mathbf{q_3} \\
&= \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \sqrt{5}\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \\ 0 \end{bmatrix} - \frac{10}{\sqrt{30}}\begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ 5/\sqrt{30} \\ 0 \end{bmatrix} - \frac{2}{\sqrt{42}}\begin{bmatrix} -1/\sqrt{42} \\ -2/\sqrt{42} \\ 1/\sqrt{42} \\ 6/\sqrt{42} \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 2/3 \\ 5/3 \\ 0 \end{bmatrix} - \begin{bmatrix} -1/21 \\ -2/21 \\ 1/21 \\ 6/21 \end{bmatrix} \\
&= \begin{bmatrix} 5/7 \\ 10/7 \\ -5/7 \\ 5/7 \end{bmatrix}
\end{aligned}$$

And we get the same result.

Yet another way is to use the formula for the projection matrix onto F. To compute the projection, we use the orthonormal basis $\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}$ rather than the original basis, as this will make the calculations easier. Letting

$$Q = \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} & \mathbf{q_3} \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{30} & -1/\sqrt{42} \\ 1/\sqrt{5} & 2/\sqrt{30} & -2/\sqrt{42} \\ 0 & 5/\sqrt{30} & 1/\sqrt{42} \\ 0 & 0 & 6/\sqrt{42} \end{bmatrix},$$

we get that the projection matrix P is

$$P = Q(Q^TQ)^{-1}Q^T = QQ^T = \frac{1}{7}\begin{bmatrix} 6 & -2 & 1 & -1 \\ -2 & 3 & 2 & -2 \\ 1 & 2 & 6 & 1 \\ -1 & -2 & 1 & 6 \end{bmatrix}.$$

Thus

$$\mathbf{p} = P\mathbf{b} = \frac{1}{7}\begin{bmatrix} 16 \\ -3 \\ 12 \\ 2 \end{bmatrix}$$

and therefore

$$\mathbf{b} - \mathbf{p} = \frac{1}{7}\begin{bmatrix} 5 \\ 10 \\ -5 \\ 5 \end{bmatrix}.$$

And again we can compute its length.

2. (8 pts.) We want to find the values of β and e that give the best overall match to the equation $r = \beta + e(r\cos\theta)$ where the values for r and $r\cos\theta$ come from our data. The matrix equation we are attempting to match is therefore

$$\begin{bmatrix} 3.0 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{bmatrix} = \beta\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + e\begin{bmatrix} 3.0\cos(0.88) \\ 2.3\cos(1.10) \\ 1.65\cos(1.42) \\ 1.25\cos(1.77) \\ 1.01\cos(2.14) \end{bmatrix} = \begin{bmatrix} 1 & 3.0\cos(0.88) \\ 1 & 2.3\cos(1.10) \\ 1 & 1.65\cos(1.42) \\ 1 & 1.25\cos(1.77) \\ 1 & 1.01\cos(2.14) \end{bmatrix}\begin{bmatrix} \beta \\ e \end{bmatrix}$$

(although of course we don't expect to be able to attain equality). Denoting the vector on the left by b, the matrix on the right hand side by A, and the vector on the right hand side by x, we are in the familiar situation of trying to get the best fit for $b = Ax$; as we know, the solution is obtained by solving $A^Tb = A^TAx$ for x. This is achieved with the following MATLAB code:

```
>> A=[1,3.0*cos(0.88);1,2.3*cos(1.10);1,1.65*cos(1.42);1,1.25*cos(1.77);
1,1.01*cos(2.14)]

A =

    1.0000    1.9115
    1.0000    1.0433
    1.0000    0.2479
    1.0000   -0.2474
    1.0000   -0.5444
```

```
>> b=[3.0;2.3;1.65;1.25;1.01]

b =

    3.0000
    2.3000
    1.6500
    1.2500
    1.0100

>> x=(A'*A)^-1*A'*b

x =

    1.4509
    0.8111
```

Hence, the eccentricity is 0.8111 and the β value is 1.4509.

3. (8 pts.)

Consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(a) We let $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{a_3}$ be the columns of A. We get

$$\mathbf{u_1} = \mathbf{a_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

We take

$$\mathbf{u_2} = \mathbf{a_2} - \frac{\mathbf{u_1}^T\mathbf{a_2}}{\mathbf{u_1}^T\mathbf{u_1}}\mathbf{u_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}.$$

We get

$$\mathbf{u_3} = \mathbf{a_3} - \frac{\mathbf{u_1}^T\mathbf{a_3}}{\mathbf{u_1}^T\mathbf{u_1}}\mathbf{u_1} - \frac{\mathbf{u_2}^T\mathbf{a_3}}{\mathbf{u_2}^T\mathbf{u_2}}\mathbf{u_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

We verify that indeed u_1, u_2 and u_3 are orthogonal.

Scaling these vectors we get:

$$\mathbf{q_1} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix},$$

$$\mathbf{q_2} = \frac{2}{\sqrt{3}}\begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 \\ \sqrt{3}/6 \\ \sqrt{3}/6 \\ \sqrt{3}/6 \end{bmatrix},$$

$$\mathbf{q_3} = \frac{\sqrt{3}}{\sqrt{2}}\begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{6}/3 \\ \sqrt{6}/6 \\ \sqrt{6}/6 \end{bmatrix}.$$

(b) We take

$$Q = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ 1/2 & \sqrt{3}/6 & -\sqrt{6}/3 \\ 1/2 & \sqrt{3}/6 & \sqrt{6}/6 \\ 1/2 & \sqrt{3}/6 & \sqrt{6}/6 \end{bmatrix}.$$

And we get R from

$$R = Q^T A = \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & \sqrt{3}/2 & \sqrt{3}/3 \\ 0 & 0 & \sqrt{6}/3 \end{bmatrix}.$$

We check that indeed $A = QR$.

4. (4 pts.) The crucial observation to make is that A is invertible (columns are linearly independent).

   (a) If $AM = 0$ then it means that $M = A^{-1}AM = A^{-1}0 = 0$, proving what we need.

   (b) For any given B, we can let $M = A^{-1}B$ and indeed we have that $AM = B$.

5. (8 pts.) We need to solve this exercise for *any* vector space V and W with a common basis $\mathbf{v_1}$, $\mathbf{v_2}$. It is not sufficient to just solve it for the case in which $V = W = R^2$. But remember we can *always* represent any linear transformation by a matrix A which tells where the basis vectors are mapped.

   (a) We can take the linear transformation T such that $T(\mathbf{v_1}) = -\mathbf{v_1}$ and $T(\mathbf{v_2}) = -\mathbf{v_2}$ (how the basis vectors are mapped uniquely determines the linear transformation). This corresponds to the matrix

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

   and indeed $A^2 = I$.

We could also have selected for example:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) We could take T such that $T(\mathbf{v_1}) = \mathbf{v_1}$ and $T(\mathbf{v_2}) = \mathbf{v_1}$. This corresponds to the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

and indeed $A^2 = A$.

There were many other choices. One is simply to take T such that $T(\mathbf{v}) = 0$ for all vectors $\mathbf{v}$.

(c) If there was a linear transformation T which can be used for both (a) and (b), it would correspond to a matrix $A \neq I$ such that $A^2 = I$ and $A^2 = A$. That means $A = I$, which is a contradiction.