18.06, Fall 2004, Problem Set 3 Solutions

1. (6 pts.)

   (a) No. The set F is not closed under scalar multiplication. For example, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is in F but $-1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ is not.

   (b) No. For a counter-example, consider $f(x) = x^2+x$; then f is in our set but $2f = 2x^2+2x$ is not.

   (c) Yes. Note that the "vectors" of this space are 4×2 matrices. If N_1 and N_2 are matrices in F, and c is any scalar, then

   $$M(N_1 + N_2) = MN_1 + MN_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

   and

   $$M(cN_1) = cMN_1 = c\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

   so $N_1 + N_2$ and cN_1 are also in F.

2. This question is not being graded. The notion of rotation was a bit ambiguous. If you consider a rotation by 0 to be the same as a rotation by 2π then this is not a vector space. Indeed, you would have for example two vectors, rotation by 0 and by π, such that if you multiply them by 2 you get the same vector.

3. (8 pts.) Each column of A is a linear combination of the columns of P, with coefficients from the correspoding column of Q:

   $$A_i = \sum_{k=1}^{p} Q_{k,i} P_k$$

   where A_i denotes the ith column of A, similarly for P_k, and as usual $Q_{k,i}$ denotes the entry of Q in row k and column i. Now if v is a vector in $C(A)$, it can be written as a linear combination of the columns of A; say

   $$v = \sum_{i=1}^{n} c_i A_i$$

   for some scalars c_i. Substituting, we get

   $$v \;=\; \sum_{i=1}^{n} c_i \left(\sum_{k=1}^{p} Q_{k,i} P_k \right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\sum_{k=1}^{p} c_i Q_{k,i} P_k \\
&= \sum_{k=1}^{p}\left(\sum_{i=1}^{n} c_i Q_{k,i}\right) P_k
\end{aligned}
$$

The point is, we now have v written as a linear combination of the columns of P. Therefore, we have shown that if v is in $C(A)$, then v is in $C(P)$, and so $C(A) \subseteq C(P)$.

It need not be the case that $C(A) = C(P)$, though. Consider for example

$$
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.
$$

Clearly $C(A) \neq C(P)$ in this case.

4. (18 pts.)

(a) Perform elimination on the first column with

$$
E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{bmatrix}
$$

to get

$$
\begin{bmatrix} 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.
$$

Now perform elimination on the third column using

$$
E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}
$$

to get

$$
\begin{bmatrix} 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
$$

(b) The pivot variables are x_1, x_3 and x_5. The free variables are x_2 and x_4.

(c) All that is required to get to reduced row echelon form is to add 2 times row 2 to row 1 (with $E_{12} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$) and divide row 3 by -2 (with $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$) to get

$$\begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(d) The first special solution is obtained by setting $x_2 = 1$ and $x_4 = 0$, from which we get $\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Setting $x_2 = 0$ and $x_4 = 1$ we get the other special solution, $\mathbf{x} = \begin{bmatrix} -5 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

(e) 3: there are 3 pivots.

(f) Note that, as long as the pivot rows and columns are included in a submatrix, row reduction on that submatrix will proceed exactly as it did for the full matrix. In particular, if we take *only* the rows and columns of A containing pivots, the resulting submatrix will have the $r \times r$ identity matrix as its reduced row echelon form. Therefore, this submatrix of A will be invertible. In our particular case, we get the submatrix

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & -3 & 0 \\ 3 & -5 & -2 \end{bmatrix}.$$

5. (8pts.) The important realisation to make for this problem is that A is the product of your MIT ID as a column vector with your MIT ID as a row vector:

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{bmatrix}$$

For a start, it makes the MATLAB code very simple!

(a) As for problem set 1, we'll give the computation for MIT ID 987654321.

```
>> a=[9;8;7;6;5;4;3;2;1]
```

```
a =

     9
     8
     7
     6
     5
     4
     3
     2
     1

>> A=a*a'

A =

    81    72    63    54    45    36    27    18     9
    72    64    56    48    40    32    24    16     8
    63    56    49    42    35    28    21    14     7
    54    48    42    36    30    24    18    12     6
    45    40    35    30    25    20    15    10     5
    36    32    28    24    20    16    12     8     4
    27    24    21    18    15    12     9     6     3
    18    16    14    12    10     8     6     4     2
     9     8     7     6     5     4     3     2     1

>> B=A+A^2+A^3

B =

  Columns 1 through 5

     6602391     5868792     5135193     4401594     3667995
     5868792     5216704     4564616     3912528     3260440
     5135193     4564616     3994039     3423462     2852885
     4401594     3912528     3423462     2934396     2445330
     3667995     3260440     2852885     2445330     2037775
     2934396     2608352     2282308     1956264     1630220
     2200797     1956264     1711731     1467198     1222665
     1467198     1304176     1141154      978132      815110
      733599      652088      570577      489066      407555

  Columns 6 through 9
```

```
     2934396     2200797     1467198      733599
     2608352     1956264     1304176      652088
     2282308     1711731     1141154      570577
     1956264     1467198      978132      489066
     1630220     1222665      815110      407555
     1304176      978132      652088      326044
      978132      733599      489066      244533
      652088      489066      326044      163022
      326044      244533      163022       81511

>> rank(B)

ans =

     1
```

(b) With the expression for A above, we can calculate B more explicitly. As in the MATLAB computation above, let us denote by $\mathbf{a}$ the column vector with entries the digits of your MIT ID. Then

$$\begin{aligned} B &= A + A^2 + A^3 \\ &= \mathbf{a}\mathbf{a}^T + \mathbf{a}\mathbf{a}^T\mathbf{a}\mathbf{a}^T + \mathbf{a}\mathbf{a}^T\mathbf{a}\mathbf{a}^T\mathbf{a}\mathbf{a}^T \\ &= \mathbf{a}\mathbf{a}^T + \mathbf{a}\|\mathbf{a}\|^2\mathbf{a}^T + \mathbf{a}\|\mathbf{a}\|^4\mathbf{a}^T \\ &= (1 + \|\mathbf{a}\|^2 + \|\mathbf{a}\|^4)\mathbf{a}\mathbf{a}^T \end{aligned}$$

Since the expression in parentheses is a scalar, the rank of B equals the rank of $\mathbf{a}\mathbf{a}^T$. Now, each column of $\mathbf{a}\mathbf{a}^T$ is just a multiple of $\mathbf{a}$, so the rank of $\mathbf{a}\mathbf{a}^T$, and therefore B, is 1 (unless you happen to have the MIT ID 000000000, in which case the rank is 0).