18.06, Fall 2004, Problem Set 1 Solutions

1. (6 pts.)

   (a) False. Counterexample: $u = (1, 0, 0)$, $v = (0, 1, 0)$ and $w = (0, 0, 1)$.

   (b) True. $u \cdot (v + 2w) = u \cdot v + 2u \cdot w = 0 + 0 = 0$.

   (c) True. $||u - v||^2 = (u - v) \cdot (u - v) = u \cdot u - 2u \cdot v + v \cdot v = 1 + 0 + 1 = 2$.

2. (6 pts.)

$$(A+B)^2 = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix}.$$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}.$$

   The correct rule is $(A + B)^2 = A^2 + AB + BA + B^2$.

3. (6 pts.)

   (a) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is such that $A^2 = 0$. We could have replaced the 1 by any other (nonzero) value as well.

   (b) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is such that $A^2 \neq 0$ and $A^3 = 0$. We could have replaced the 1's by other values as well.

4. (10 pts.) Eliminating element $(2, 1)$ using $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we get:

$$\begin{cases} x & +2y & +3z & = & 0 \\ & 3y & +z & = & -3 \\ 2x & +y & +z & = & 3 \end{cases}$$

   Eliminating element $(3, 1)$ using $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, we get:

$$\begin{cases} x & +2y & +3z & = & 0 \\ & 3y & +z & = & -3 \\ & -3y & -5z & = & 3 \end{cases}$$

   Eliminating element $(3, 2)$ using $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, we get:

$$\begin{cases} x & +2y & +3z & = & 0 \\ & 3y & +z & = & -3 \\ & & -4z & = & 0 \end{cases}$$

After back substitution, we get $z = 0$, $y = -1$ and $x = 2$.

Now,

$$\begin{aligned} M &= E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}. \end{aligned}$$

The product $E_{31}E_{21}$ was very simple to compute (the product has the second row of E_{21} and the third row of E_{31}) since adding a multiple of row 1 to row 2 and adding a multiple of row 1 to row 3 can be done independently without affecting the result.

5. (6 pts.)

   - E_{21} and E_{31} commute ($E_{21}E_{31} = E_{31}E_{21}$) since adding a multiple of row 1 to row 2 and adding a multiple of row 1 to row 3 can be done in any order without affecting the result.
   - E_{21} and E_{32} do not commute. $E_{21}E_{32}B$ (for any matrix B means adding a multiple of row 2 of B to row 3 and then adding a multiple of row 1 to row 2. This is not the same (at least for some B) as first adding a multiple of row 1 to row 2 and then a multiple of row 2 (which has just changed) to row 3. Thus, $E_{21}E_{32}B \neq E_{32}E_{21}B$ for some B and thus $E_{21}E_{32} \neq E_{32}E_{21}$.
   - E_{31} and E_{32} commute since adding a multiple of row 1 to row 3 and a multiple of row 2 to row 3 can be done in any order without affecting the result.

6. (6 pts.) Independently of your MIT ID, the answer will be very close to the 3×3 matrix with all entries equal to $1/3$. We will see why in section 8.3 later in the semester.

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>> a=9; b=8; c=7; d=6; e=5; f=4; g=3; h=2; i=1;
>> A=[a+b+c e+g+h d+f+i; d+e+f a+c+i b+g+h; g+h+i b+d+f a+c+e]

A =

    24    10    11
    15    17    13
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     6    18    21

>> B=A/(a+b+c+d+e+f+g+h+i)

B =

    0.5333    0.2222    0.2444
    0.3333    0.3778    0.2889
    0.1333    0.4000    0.4667

>> B^40 =

    0.3333    0.3333    0.3333
    0.3333    0.3333    0.3333
    0.3333    0.3333    0.3333
```